Chapter 5

Waves I: Generalities, Superposition & Standing Waves

5.1 The Important Stuff

5.1.1 Wave Motion

Wave motion occurs when the mass elements of a medium such as a taut string or the surface of a liquid make relatively small oscillatory motions but *collectively* give a pattern which travels for long distances. This kind of motion also includes the phenomenon of *sound*, where the molecules in the air around us make small oscillations but collectively give a disturbance which can travel the length of a college classroom, all the way to the students dozing in the back. We can even view the up–and–down motion of inebriated spectators of sports events as wave motion, since their small *individual* motions give rise to a disturbance which travels around a stadium.

The mathematics of wave motion also has application to electromagnetic waves (including visible light), though the physical origin of *those* traveling disturbances is quite different from the *mechanical* waves we study in this chapter; so we will hold off on studying electromagnetic waves until we study electricity and magnetism in the second semester of our physics course.

Obviously, wave motion is of great importance in physics and engineering.

5.1.2 Types of Waves

In some types of wave motion the motion of the elements of the medium is (for the most part) *perpendicular* to the motion of the traveling disturbance. This is true for waves on a string and for the people–wave which travels around a stadium. Such a wave is called a **transverse wave**. This type of wave is the easiest to visualize.

For other waves the motion of the elements of the medium is *parallel* to the motion of the disturbance. This type of wave can be seen when we stretch a spring and wiggle its end parallel to its length. We then see traveling regions where the spring is more compressed and more stretched; the elements of the spring have a small back–and–forth motion. A sound wave travels in the same way; here, the air molecules have a small back-and-forth motion generating regions where the air has greater or smaller compression.

A wave which travels due to local motions *along* the direction of propagation is called a **longitudinal wave**.

5.1.3 Mathematical Description of a Wave; Wavelength, Frequency and Wave Speed

In our mathematical treatment of wave phenomena, we will mostly deal with one-dimensional traveling waves. The coordinate along which the disturbance travels will be x; at each value of x the medium will be "displaced" in some way, and that displacement will be described by the variable y. Then y will depend upon x (the *place* where we see the displacement of the medium) and also the time t at which we see the displacement. In general, $y = f(x,t)$.

If we specialize to the case where the shape of the wave does not change with time, but rather travels along the $+x$ axis with some velocity v, then the wave will be a function *only* of the combination $x - vt$:

$$y = f(x - vt) \qquad \text{(Velocity } v \text{ in } +x \text{ direction)} .$$

Of course, from this it follows that a wave travelling in the other direction is

$$y = f(x + vt) \qquad \text{(Velocity } v \text{ in } -x \text{ direction)} .$$

For reasons which are basically mathematical, it is important to study a *particular* traveling wave, one which has a sinusoidal shape as a function of x. Such a wave is given by

$$y(x,t) = y_m \sin(kx \mp \omega t) \tag{5.1}$$

In this equation, the $-$ sign is chosen for a wave traveling in the $+x$ direction; $+$ is chosen for a wave traveling in the $-x$ direction.

This wave form represents an (infinite) *wave train* rather than a pulse. In this formula, k is called the **angular wave number** and it has units of m^{-1}. ω is called the **angular frequency** for the wave (as you would expect!) and it has units of s^{-1}.

The harmonic wave of Eq. 5.1 is periodic in both space and time. The **wavelength** λ of the wave is the distance between repetitions of the (sinusoidal) wave shape when we "freeze" the wave in time. One can show that it is related to k by:

$$k = \frac{2\pi}{\lambda} . \tag{5.2}$$

The **period** T of the wave is the time between repetitions of the motion of any one element of the medium. One can show that it is related to the angular frequency by:

$$\omega = \frac{2\pi}{T} . \tag{5.3}$$

As in our study of oscillations, the **frequency** of the wave has the same relation to ω and T:

$$f = \frac{\omega}{2\pi} = \frac{1}{T}$$

One can show that the speed of such a wave (that is, the rate at which its crests travel along the x axis) is

$$v = \frac{\omega}{k} = \lambda f \tag{5.4}$$

One must be careful not to confuse the speed of the *wave* with the speeds of the individual *elements* of the medium through which the wave passes. Each mass point of the medium moves like a harmonic oscillator whose amplitude and frequency is the same as that of the wave; the maximum speed of each element is $v_{\rm max} = y_m\omega$ and the maximum acceleration of each element is $y_m\omega^2$.

5.1.4 Waves on a Stretched String

One of the simplest examples of a transverse wave is that of waves travelling on a stretched string. One can show that for a string under a tension τ whose mass per unit length is given by μ, the speed of waves is

$$v = \sqrt{\frac{\tau}{\mu}} \tag{5.5}$$

An important feature of waves is that they *transmit energy*. The **average power** is the rate at which mechanical energy passes any point of the x axis. One can show that for harmonic waves on a string it is given by

$$\overline{P} = \tfrac{1}{2}\mu v \omega^2 y_m^2 \tag{5.6}$$

where μ is the linear mass density of the string, v is the speed of string waves and ω is the angular frequency of the harmonic wave. We note here that $\overline{P}$ is proportional to the *squares* of ω and y_m.

5.1.5 The Principle of Superposition

A simple (but non-obvious!) property of all of the waves in nature that we are likely to study is that they *add together*. More precisely, if one physical disturbance of a medium generates the wave $y_1(x,t)$ and another disturbance generates the wave $y_2(x,t)$ then if *both* effects act at the same time, the *resultant wave* will be

$$y_{\rm Tot}(x,t) = y_1(x,t) + y_2(x,t)\ . \tag{5.7}$$

5.1.6 Interference of Waves

In this and the next section we give the results for the superposition of two harmonic waves which differ in only one respect; this allows us to understand the importance that the parts of Eq. 5.1 play in the combining of waves.

First we take two waves with the same speed, frequency, amplitude and direction of motion but which differ by a phase constant. We will combine the two waves

$$y_1 = y_m \sin(kx - \omega t) \qquad \text{and} \qquad y_2 = y_m \sin(kx - \omega t + \phi) \tag{5.8}$$

To arrive at a useful form for the sum of these two waves one can use the trig identity

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \ .$$

With this, we can show that the resultant wave $y'(x,t)$ is:

$$\begin{aligned} y'(x,t) &= y_1(x,t) + y_2(x,t) \\ &= \left[2y_m \cos\left(\tfrac{1}{2}\phi\right)\right] \sin(kx - \omega t + \tfrac{1}{2}\phi) \end{aligned} \tag{5.9}$$

The resultant wave has a new phase but more importantly it has a new amplitude which depends on y_m and ϕ:

$$y'_m = 2y_m \cos\tfrac{1}{2}\phi \ . \tag{5.10}$$

When $\phi = 0$, the new amplitude is $2y_m$; the waves are said to be **completely in phase** and that the addition is **fully constructive**. A maximum from wave y_1 coincides with a maximum from wave y_2 and a “bigger” wave is the result.

When $\phi = \pi$ then the new amplitude is *zero* and the waves are said to be **completely out of phase** and that the addition is **fully destructive**. Here a maximum from wave y_1 coincides with a *minimum* of wave y_2 and the result is complete cancellation.

5.1.7 Standing Waves

Next we consider the result of the addition of two harmonic waves which have the same speed, frequency and amplitude but for which the directions of propagation (either $+x$ or $-x$ are different. (In this case the phase constants for each wave won’t matter.) We will add:

$$y_1(x,t) = y_m \sin(kx - \omega t) \qquad \text{and} \qquad y_2(x,t) = y_m \sin(kx + \omega t) \tag{5.11}$$

One again we can use the trig identity for adding two sines to show that the sum $y'_m(x,t)$ is:

$$\begin{aligned} y'_m(x,t) &= y_1(x,t) + y_2(x,t) \\ &= [2y_m \cos\omega t] \sin kx \end{aligned} \tag{5.12}$$

The resultant wave in Eq. 5.12 is a very interesting function of x and t, though it is *not* a travelling wave since it is not of the form $f(kx \mp \omega t)$. It is a sinusoidal function of the coordinate x, multiplied by a modulating factor $\cos(\omega t)$. Since the same spatial pattern of oscillations stays in one place, a wave of the form of Eq. 5.12 is called a **standing wave**.

For the wave given in Eq. 5.12 there are points where there is no displacement, i.e. those where $\sin(kx) = 0$ (so that $x = \frac{n\pi}{k}$ with n equal to an integer). These points are the **nodes** of the standing wave pattern.

There are also points for which the displacement is a maximum, namely those for which $\sin(kx) = \pm 1$ (so that $x = \frac{(2n+1)\pi}{2k}$ with n equal to an integer). These points are called the **antinodes** of the standing wave pattern. Consecutive nodes and antinodes are separated by $\lambda/2$, where λ is the wavelength of the original waves that went into making up the standing wave. A node and the closest antinode are separated by $\lambda/4$.

5.1.8 Standing Waves on Strings Under Tension

We can observe standing wave patterns in the lab if we clamp the ends of a piece of string (putting it under some tension) and pluck it, or possibly give one end of the string a small jiggly motion of a certain frequency. In the first case it is mostly the fundamental mode ($n = 1$) of vibration which is set up on the string, whereas in the second case we can selectively make any mode oscillate on the string.

The **oscillation modes** of the string are those where the string oscillates with nodes at either end and a special pattern of nodes and antinodes in between. Such a pattern exists only when the string vibrates with certain **resonant frequencies**. For these modes, the length of the string L is an integer number of half–wavelengths:

$$L = n\frac{\lambda}{2} \qquad n = 1, 2, 3, \ldots$$

which leads to the formula for the resonant frequencies:

$$f_n = n\frac{v}{2L} \qquad n = 1, 2, 3, \ldots \tag{5.13}$$

where v is the speed of waves on the string and f_n is the resonant frequency of the n^{th} mode.

5.2 Worked Examples

5.2.1 Wavelength, Frequency and Speed

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1. A wave has speed $240\,\frac{\mathrm{m}}{\mathrm{s}}$ and a wavelength of $3.2\,\mathrm{m}$. What are the (a) frequency and (b) period of the wave? [HRW5 17-1]

(a) From Eq. 5.4 we have

$$v = \lambda f \qquad \Longrightarrow \qquad f = \frac{v}{\lambda} \; .$$

Plug in the given v and λ:

$$f = \frac{(240\,\frac{\mathrm{m}}{\mathrm{s}})}{(3.2\,\mathrm{m})} = 75\,\mathrm{s}^{-1} = 75\,\mathrm{Hz}$$

(b) Then, from $T = 1/f$, we get the period of the wave:

$$T = \frac{1}{f} = \frac{1}{75\,\mathrm{s}^{-1}} = 1.3 \times 10^{-2}\,\mathrm{s} = 13\,\mathrm{ms}$$

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2. Write the equation for a (harmonic) wave travelling in the negative direction along the x axis and having an amplitude of $0.010\,\mathrm{m}$, a frequency of $550\,\mathrm{Hz}$, and a speed of $330\,\frac{\mathrm{m}}{\mathrm{s}}$. [HRW5 17-5]

From Eq. 5.4 we get the wavelength for this wave:

$$\lambda = \frac{v}{f} = \frac{(330\,\frac{\mathrm{m}}{\mathrm{s}})}{(550\,\mathrm{s}^{-1})} = 0.600\,\mathrm{m}$$

and we get the angular wave number k and the angular frequency ω:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{(0.600\,\mathrm{m})} = 10.5\,\mathrm{m}^{-1}$$

$$\omega = 2\pi f = 2\pi(550\,\mathrm{s}^{-1}) = 3.46 \times 10^3\,\mathrm{s}^{-1}$$

This gives us all the quantities we need to put into the harmonic wave of Eq. 5.1. Using the given amplitude $y_m = 0.010\,\mathrm{m}$ and choosing the proper sign for a wave traveling in the $-x$ direction, we get:

$$y(x,t) = (0.010\,\mathrm{m})\sin\Big([10.5\,\mathrm{m}^{-1}]x + [3.46 \times 10^3\,\mathrm{s}^{-1}]t\Big)$$

5.2.2 Waves on a Stretched String

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3. The speed of a transverse wave on a string is $170\,\frac{\mathrm{m}}{\mathrm{s}}$ when the string tension is $120\,\mathrm{N}$. To what value must the tension be changed to raise the wave speed to $180\,\frac{\mathrm{m}}{\mathrm{s}}$? [HRW5 17-19]

Knowing the speed of waves on a string for any particular tension allows us to find the mass density μ of the string. Use:

$$v = \sqrt{\frac{F}{\mu}} \qquad \Longrightarrow \qquad v^2 = \frac{F}{\mu} \qquad \Longrightarrow \qquad \mu = \frac{F}{v^2}$$

Plug in the numbers for the first case:

$$\mu = \frac{(120\,\mathrm{N})}{(170\,\frac{\mathrm{m}}{\mathrm{s}})^2} = 4.15 \times 10^{-3}\,\frac{\mathrm{kg}}{\mathrm{m}}$$

Now for some *new* tension the speed of the waves on the string is $180\,\frac{\mathrm{m}}{\mathrm{s}}$. It's the same string so the mass density is still $4.15 \times 10^{-3}\,\frac{\mathrm{kg}}{\mathrm{m}}$. Then find the new tension F:

$$F = \mu v^2 = (4.15 \times 10^{-3}\,\tfrac{\mathrm{kg}}{\mathrm{m}})(180\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 135\,\mathrm{N}$$

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4. A string along which waves can travel is $2.70\,\mathrm{m}$ long and has a mass of $260\,\mathrm{g}$. The tension is the string is $36.0\,\mathrm{N}$. What must be the frequency of travelling waves of amplitude $7.70\,\mathrm{mm}$ for the average power to be $85.0\,\mathrm{W}$? [HRW6 17-24]

A couple properties of this string which we can calculate: We can find its linear mass density:

$$\mu = \frac{M}{L} = \frac{(260 \times 10^{-3}\,\mathrm{kg})}{(2.70\,\mathrm{m})} = 9.63 \times 10^{-2}\,\tfrac{\mathrm{kg}}{\mathrm{m}}$$

and also the speed of waves on the string:

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{(36.0\,\mathrm{N})}{(9.63 \times 10^{-2}\,\frac{\mathrm{kg}}{\mathrm{m}})}} = 19.3\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Using Eq. 5.6 to relate the rate of energy transmission $\overline{P}$ to the other quantities, the angular frequency of the travelling wave is

$$\begin{aligned}
\omega^2 &= \frac{2\overline{P}}{\mu v y_m^2} \\
&= \frac{2(85.0\,\mathrm{W})}{(9.63 \times 10^{-2}\,\frac{\mathrm{kg}}{\mathrm{m}})(19.3\,\frac{\mathrm{m}}{\mathrm{s}})(7.70 \times 10^{-3}\,\mathrm{m})} = 1.53 \times 10^{6}/\,\mathrm{s}^2
\end{aligned}$$

so that

$$\omega = 1.24 \times 10^3/\,\mathrm{s}$$

and then

$$f = \frac{\omega}{2\pi} = \frac{(1.24 \times 10^3/\,\mathrm{s})}{2\pi} = 198\,\mathrm{Hz}$$

Waves of the given amplitude must have frequency 198 Hz.

5.2.3 Superposition; Interference of Waves

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5. Two identical travelling waves, moving in the same direction, are out of phase by $\frac{\pi}{2}$ rad. What is the amplitude of the resultant wave in terms of the common amplitude y_m of the two combining waves? [HRW5 17-36]

Though the answer is given quite simply by Eq. 5.10 with $\phi = \frac{\pi}{2}$, we'll work out all the steps here.

We will make specific choices for the mathematical forms for the two waves, but one can show that the answer doesn't depend on the choice.

Suppose the first wave has wave number k and angular frequency ω and travels in the $+x$ direction. Then one choice for this wave is

$$y_1(x,t) = y_m \sin(kx - \omega t) \tag{5.14}$$

Now if the second wave is "identical" it must have the same amplitude (y_m). It will also be a sine wave whose (x, t) dependence also has the form $kx - \omega t$. But the argument of the sin function — that is, the "phase" — differs from that of y_1 by $\frac{\pi}{2}$. The following choice:

$$y_2(x,t) = y_m \sin(kx - \omega t - \tfrac{\pi}{2}) \tag{5.15}$$

will put wave y_2 *ahead* of y_1 in space by a quarter wave (since at any given time t we need to increase the term kx in y_2 by $\frac{\pi}{2}$ in order to give the same answer as y_1). But again, the opposite choice of sign for the phase difference will give the same result.

When waves y_1 and y_2 are travelling on the same string, the resultant wave is the sum of 5.14 and 5.15,

$$\begin{aligned} y_{\rm Tot}(x,t) &= y_1(x,t) + y_2(x,t) \\ &= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t - \tfrac{\pi}{2}) \end{aligned} \tag{5.16}$$

Here, the common factor y_m can taken from both terms and for the rest we can use the trig identity

$$\sin\alpha + \sin\beta = 2\sin\left[\tfrac{1}{2}(\alpha+\beta)\right]\cos\left[\tfrac{1}{2}(\alpha-\beta)\right]$$

where we intend to use

$$\alpha = kx - \omega t \qquad \text{and} \qquad \beta = kx - \omega t - \tfrac{\pi}{2}$$

Putting all of this into Eq. 5.16 gives

$$\begin{aligned} y_{\rm Tot}(x,t) &= 2y_m \sin\left[\tfrac{1}{2}(2kx - 2\omega t - \tfrac{\pi}{2})\right]\cos\left[\tfrac{1}{2}(\tfrac{\pi}{2})\right] \\ &= 2y_m \sin\left[\tfrac{1}{2}(kx - \omega t - \tfrac{\pi}{4})\right]\cos\left[\tfrac{\pi}{4}\right] \\ &= \sqrt{2}y_m \sin\left[\tfrac{1}{2}(kx - \omega t - \tfrac{\pi}{4})\right] \end{aligned}$$

where in the last step we have used the well–known fact that $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

Our final result for $y_{\rm Tot}(x,t)$ is a sinusoidal function of x and t which has amplitude

$$y_{\rm Tot,\,m} = \sqrt{2}y_m \approx 1.41 y_m \ ,$$

that is, it is $\sqrt{2}$ times the amplitude of either of the original waves.

5.2.4 Standing Waves on a Stretched String

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6. A nylon guitar string has a linear density of $7.2\ \mathrm{g/m}$ and is under a tension of $150\,\mathrm{N}$. The fixed supports are $90\,\mathrm{cm}$ apart. The string is oscillating in the standing wave pattern shown in Fig. 5.1. Calculate the (a) speed, (b) wavelength, and (c) frequency of the travelling waves whose superposition gives this standing wave. [HRW6 17-33]

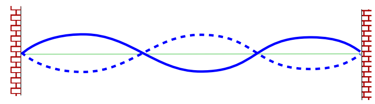

Figure 5.1: Standing wave on pattern on string of Example 6

(a) We have the linear mass density $\mu = 7.2 \times 10^{-3}\,\frac{\mathrm{kg}}{\mathrm{m}}$ and the tension $\tau = 150\,\mathrm{N}$ for the string. From Eq. 5.5, the speed of waves on the string is

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{(150\,\mathrm{N})}{(7.2 \times 10^{-3}\,\frac{\mathrm{kg}}{\mathrm{m}})}} = 140\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

(b) For the pattern shown in Fig. 5.1, the length of the sring is equal to 3 half-wavelengths: $L = 3(\frac{\lambda}{2})$. So the wavelength (of the travelling waves giving the pattern) is:

$$\lambda = \tfrac{2}{3}L = \tfrac{2}{3}(0.90\,\mathrm{m}) = 0.60\,\mathrm{m} = 60\,\mathrm{cm}$$

(c) For the frequency of the waves, use $\lambda f = v$. Then:

$$f = \frac{v}{\lambda} = \frac{(140\,\frac{\mathrm{m}}{\mathrm{s}})}{(0.60\,\mathrm{m})} = 240\,\mathrm{Hz}$$

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7. A string that is stretched between fixed supports separated by $75.0\,\mathrm{cm}$ has resonant frequencies of $420\,\mathrm{Hz}$ and $315\,\mathrm{Hz}$ with no intermediate frequencies. What are (a) the lowest resonant frequency and (b) the wave speed? [HRW6 17-39]

(a) The problem gives two *successive* frequencies, but we don't (yet) know what values of n in Eq. 5.13 they correspond to. What does Eq. 5.13 tell us about the difference between two successive resonant frequencies? The difference between the frequencies corresponding to $N + 1$ and n is:

$$f_{n+1} - f_n = (n+1)\frac{v}{2L} - n\frac{v}{2L} = \frac{v}{2L}$$

so that the difference does not depend on n, and it gives us $\frac{v}{2L}$. Using our frequencies, we get:

$$f_{n+1} - f_n = 420\,\mathrm{Hz} - 315\,\mathrm{Hz} = 105\,\mathrm{Hz} = \frac{v}{2L}$$

and since we have the value of L, we find:

$$v = 2L(105\,\mathrm{Hz}) = 2(0.750\,\mathrm{m})(105\,\mathrm{Hz}) = 158\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

which is the answer to part (b)! Anyway, the lowest frequency has $n = 1$; with this, $f_1 = \frac{v}{2L}$ which we already have found to be equal to $105\,\mathrm{Hz}$. So the fundamental frequency of the string is

$$f_1 = 105\,\mathrm{Hz}$$

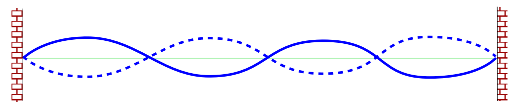

Figure 5.2: Standing wave on string as described in Example 8

(b) Answer in part (a): $v = 158\,\frac{\mathrm{m}}{\mathrm{s}}$.

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8. In an experiment on standing waves, a string $90\,\mathrm{cm}$ long is attached to the prong of an electrically driven tuning fork that oscillates perpendicular to the length of the string at a frequency of $60\,\mathrm{Hz}$. The mass of the string is $0.044\,\mathrm{kg}$. What tension must the string be under (weights are attached to the other end) if it is to oscillate in four loops? [HRW6 17-44]

The problem is telling us that the standing wave on the string has the pattern ("four loops") shown in Fig. 5.2. Thus the full length of the string is *four* half-wavelengths, so that:

$$L = 4\left(\frac{\lambda}{2}\right) \qquad \Longrightarrow \qquad \lambda = \frac{L}{2} = \frac{(0.90\,\mathrm{m}}{2} = 0.45\,\mathrm{m}$$

The frequency of the wave is $f = 60\,\mathrm{Hz}$ so that the speed of wave on this string is

$$v = \lambda f = (0.45\,\mathrm{m})(60\,\mathrm{Hz}) = 27\,\frac{\mathrm{m}}{\mathrm{s}}$$

The linear mass density of the string is

$$\mu = \frac{(0.044\,\mathrm{kg})}{(0.90\,\mathrm{m})} = 4.9 \times 10^{-2}\,\frac{\mathrm{kg}}{\mathrm{m}}$$

So using Eq. 5.5 for the speed of waves on a string, we get:

$$v^2 = \frac{\tau}{\mu} \qquad \Longrightarrow \qquad \tau = \mu v^2 = (4.9 \times 10^{-2}\,\tfrac{\mathrm{kg}}{\mathrm{m}})(27\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 36\,\mathrm{N}\ .$$

The tension of the string is $36\,\mathrm{N}$.