Chapter 4

Oscillatory Motion

4.1 The Important Stuff

4.1.1 Simple Harmonic Motion

In this chapter we consider systems which have a motion which repeats itself in time, that is, it is **periodic**. In particular we look at systems which have some coordinate (say, x) which has a sinusoidal dependence on time. A graph of x vs. t for this kind of motion is shown in Fig. 4.1. Suppose a particle has a periodic, sinusoidal motion on the x axis, and its motion takes it between $x = +A$ and $x = -A$. Then the general expression for $x(t)$ is

$$x(t) = A\cos(\omega t + \phi) \tag{4.1}$$

A is called the **amplitude** of the motion. For reasons which will become clearer later, ω is called the **angular frequency**. We say that a mass which has a motion of the type given in Eq. 4.1 undergoes **simple harmonic motion**.

From 4.1 we see that when the time t increases by an amount $\frac{2\pi}{\omega}$, the argument of the cosine increases by 2π and the value of x will be the same. So the motion *repeats itself* after a time interval $\frac{2\pi}{\omega}$, which we denote as T, the **period** of the motion. The number of

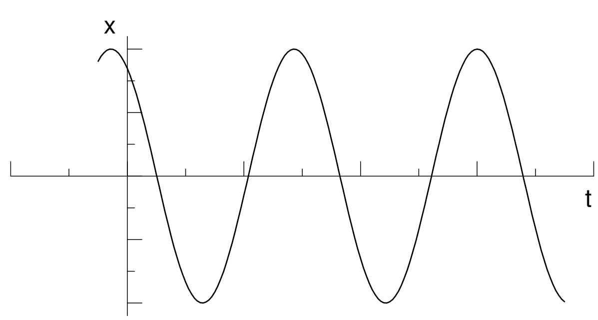

Figure 4.1: Plot of x vs. t for simple harmonic motion. (t and x axes are unspecified!)

oscillations per time is given by $f = \frac{1}{T}$, called the **frequency** of the motion:

$$T = \frac{2\pi}{\omega} \qquad\qquad f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{4.2}$$

Rearranging we have a formula for ω in terms of f or T:

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{4.3}$$

Though ω (angular frequency) and f (frequency) are closely related (with just a factor of 2π between them, we need to be careful to distinguish them; to help in this, we normally express ω in units of $\frac{\mathrm{rad}}{\mathrm{s}}$ and f in units of $\frac{\mathrm{cycle}}{\mathrm{s}}$, or Hz (Hertz). However, the real dimensions of both are $\frac{1}{\mathrm{s}}$ in the SI system.

From $x(t)$ we get the velocity of the particle:

$$v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \tag{4.4}$$

and its acceleration:

$$a(t) = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) \tag{4.5}$$

We note that the maximum values of v and a are:

$$v_{\max} = \omega A \qquad\qquad a_{\max} = \omega^2 A \tag{4.6}$$

The maximum speed occurs in the *middle* of the oscillation. (The slope of x vs. t is greatest in size when $x = 0$.) The magnitude of the acceleration is greatest at the *ends* of the oscillation (when $x = \pm A$).

Comparing Eq. 4.5 and Eq. 4.1 we see that

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{4.7}$$

which is the same as $a(t) = -\omega^2 x(t)$. Using 4.1 and 4.4 and some trig we can also arrive at a relation between the speed $|v(t)|$ of the mass and its coordinate $x(t)$:

$$\begin{aligned} |v(t)| &= \omega A |\sin(\omega t + \phi)| = \omega A \sqrt{1 - \cos^2(\omega t + \phi)} \\ &= \omega A \sqrt{1 - \left(\frac{x(t)}{A}\right)^2} \,. \end{aligned} \tag{4.8}$$

We could also arrive at this relation using energy conservation (as discussed below). Note, if we are given x we can *only* give the absolute value of v since there are *two* possibilities for velocity at each x (namely a $\pm$ pair).

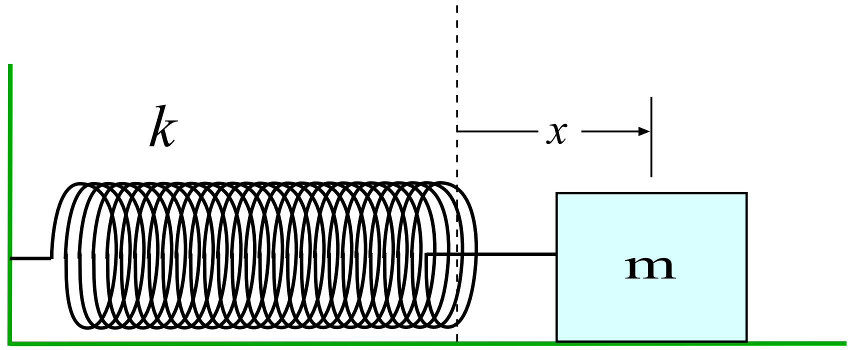

Figure 4.2: Mass m is attached to horizontal spring of force constant k; it slides on a frictionless surface!

4.1.2 Mass Attached to a Spring

Suppose a mass m is attached to the end of a spring of force constant k (whose other end is fixed) and slides on a frictionless surface. This system is illustrated in Fig. 4.2. Then if we measure the coordinate x of the mass from the place where it would be if the spring were at its equilibrium length, Newton's 2$^{\rm nd}$ law gives

$$F_x = -kx = ma_x = m\frac{d^2x}{dt^2} \ ,$$

and then we have

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \ . \tag{4.9}$$

Comparing Eqs. 4.9 and 4.7 we can identify ω^2 with $\frac{k}{m}$ so that

$$\omega = \sqrt{\frac{k}{m}} \tag{4.10}$$

From the angular frequency ω we can find the period T and frequency f of the motion:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad\qquad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{4.11}$$

It should be noted that ω (and hence T and f) does not depend on the amplitude A of the motion of the mass. In reality, of course if the motion of the mass is too large then then spring will not obey Hooke's Law so well, but as long as the oscillations are "small" the period is the same for all amplitudes.

In the lab, it's much easier to work with a mass bobbing up and down on a *vertical* spring. One can (and should!) ask if we can still use the same formulae for T and f, or if gravity (g) enters in somehow. In fact, the same formulae (Eq. 4.11) *do* apply in this case.

To be more clear about the vertical mass–spring system, we show such a system in Fig. 4.3. In (a), the spring is oriented vertically and has some *unstretched* length. (We are ignoring the mass of the spring.) When a mass m is attached to the end, the system will be

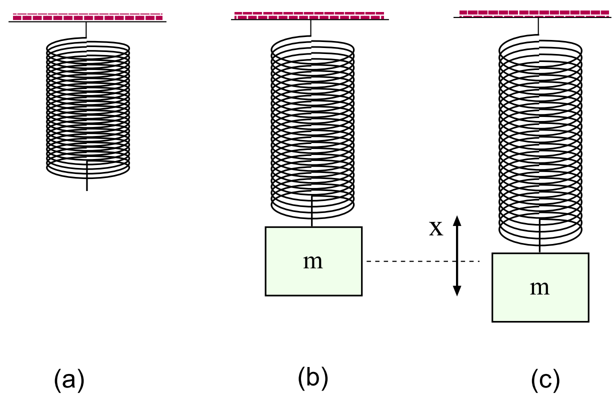

Figure 4.3: (a) Unstretched vertical spring of force constant k (assumed massless). (b) Mass attached to spring is at equilibrium when the spring has been extended by a distance mg/k. (c) Mass will undergo small oscillations about the *new* equilibrium position.

at equilibrium when the spring has been extended by some length y; balancing forces on the mass, this extension is given by:

$$ky = mg \qquad \Longrightarrow \qquad y = \frac{mg}{k} \ .$$

When the mass is disturbed from its equilibrium position, it will undergo harmonic oscillations which can be described by some coordinate x, where x is measured from the *new* equilibrium position of the end of the spring. Then the motion is just like that of the horizontal spring.

Finally, we note that for more precise work with a *real* spring–mass system one *does* need to take into account the mass of the spring. If the spring has a total mass m_s, one can show that Eq. 4.10 should be modified to:

$$\omega = \sqrt{\frac{k}{m + \frac{m_s}{3}}} \tag{4.12}$$

That is, we replace the value of the mass m by m plus *one–third* the spring's mass.

4.1.3 Energy and the Simple Harmonic Oscillator

For the mass–spring system, the kinetic energy is given by

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \tag{4.13}$$

and the potential energy is

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \cos^2(\omega t + \phi) \ . \tag{4.14}$$

Using $\omega^2 = \frac{k}{m}$ in 4.13 we then find that the total energy is

$$E = K + U = \tfrac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

and the trig identity $\sin^2\theta + \cos^2\theta = 1$ gives

$$E = \tfrac{1}{2}kA^2 \tag{4.15}$$

showing that the energy of the simple harmonic oscillator (as typified by a mass on a spring) is constant and is equal to the potential energy of the spring when it is maximally extended (at which time the mass is motionless).

It is useful to use the principle of energy conservation to derive some general relations for 1–dimensional harmonic motion. (We will not use the particular parameters for the mass–spring system, just the quantities contained in Eq. 4.1, which describes the motion of a mass m along the x axis. From Eq. 4.13 we have the kinetic energy as a function of time

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi)$$

Now the maximum value of the kinetic energy is $\frac{1}{2}m\omega^2A^2$, which occurs when $x = 0$. Since we are free to fix the "zero–point" of the potential energy, we can agree that $U(x) = 0$ at $x = 0$. Then the total energy of the system must be equal to the maximum (i.e. $x = 0$ value of the kinetic energy:

$$E = \tfrac{1}{2}m\omega^2A^2$$

Then using these expressions, the potential energy of the system is

$$\begin{aligned} U &= E - K \\ &= \tfrac{1}{2}m\omega^2A^2 - \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi) = \tfrac{1}{2}m\omega^2A^2(1 - \sin^2(\omega t + \phi)) \\ &= \tfrac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi) \\ &= \tfrac{1}{2}m\omega^2x^2 \end{aligned}$$

Of course, for the mass–spring system U is given by $\frac{1}{2}kx^2$, which gives the relation $m\omega^2 = k$, or $\omega = \sqrt{\frac{k}{m}}$, which we've already found. If we use the relation $v_{\rm max} = \omega A$ then the potential energy can be written as

$$U(x) = \tfrac{1}{2}m\omega^2x^2 = \tfrac{1}{2}\frac{mv_{\rm max}^2}{A^2}x^2 \tag{4.16}$$

4.1.4 Relation to Uniform Circular Motion

There is a *correspondence* between simple harmonic motion and uniform circular motion, which is illustrated in Fig. 4.4 (a) and (b). In (a) a mass point moves in a horizontal circular path with uniform circular motion at a radius R (for example, it might be glued to the edge of a spinning disk of radius R). Its angular velocity is ω, so its location is given by the time–varying angle θ, where

$$\theta(t) = \omega t + \phi$$

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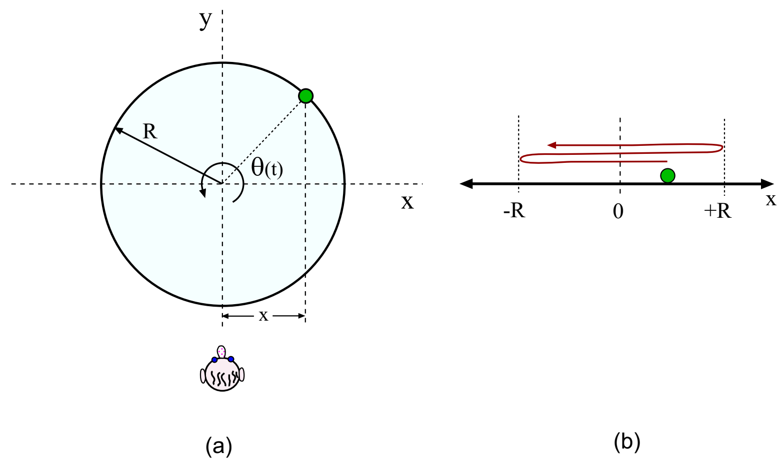

Figure 4.4: (a) Mass point moves in a horizontal circle of radius R. The *angular velocity* of its motion is ω. A guy with a big nose (seen from above) is observing the motion of the mass at the level of the circle. He sees only the x coordinate of the point's motion. (b) Motion of the mass as seen by the guy with the big nose. The *projection* of the motion is the same as simple harmonic motion with *angular frequency* ω and amplitude R.

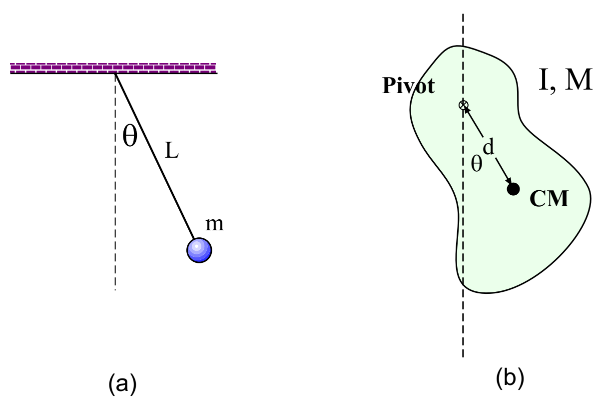

Figure 4.5: (a) Simple pendulum. (b) Physical pendulum.

In 4.4 (b) we show the motion of the mass as it would be seen by someone looking toward the $+y$ direction at the level of the disk. Such an observer sees only the changing x coordinate of the mass's motion. Since $x = R\cos\theta$, the observed coordinate is

$$x(t) = R\cos(\theta(t)) = R\cos(\omega t + \phi) \ ,$$

the same as Eq. 4.1. The motion of the corresponding (projected) harmonic oscillator has an *angular frequency* of ω and an amplitude of R.

4.1.5 The Pendulum

We start with the **simple pendulum**, which has just a small mass m hanging from a string of length L whose mass we can ignore. (See Fig. 4.5 (a).) The mass is set into motion so that it moves in a vertical plane. One can show that if θ is the angle which the string makes with the vertical, it obeys the differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

One should note that this is *not* of the form given in Eq. 4.7.

Things are much simpler when we restrict θ to be "small" at all times. If that is the case, then we can use the approximation $\sin\theta \approx \theta$, which is true if we are measuring θ in radians. Then the differential equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \tag{4.17}$$

Comparison of this equation with Eq. 4.7 lets us identify the angular frequency of the motion:

$$\omega = \sqrt{\frac{g}{L}} \tag{4.18}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \qquad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{4.19}$$

The (perhaps) surprising thing about Eqs. 4.18 and 4.19 is that they have no dependence on the mass suspended from the string or on the amplitude of the swing... as long as it is a small angle!

$$\theta(t) = \theta_{\mathrm{max}}\cos(\omega t + \phi) \tag{4.20}$$

We must always keep our assumption of "small" θ in the back of our minds whenever we do a problem with a pendulum. The formulae giving T and f become less accurate as θ_{max} gets too big.

An important generalization of the simple pendulum is that of a rigid body which is free to rotate in a plane about some (frictionless!) pivot. Such a system is known as a **physical pendulum** and is diagrammed in Fig. 4.5 (b).

Suppose we look at the line which joins the pivot to the center of mass of the object. If θ is the angle which this line makes with the vertical, and if we again use the approximation $\sin\theta \approx \theta$, one can show that it obeys the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{Mgd}{I}\theta \tag{4.21}$$

where d is the distance between the pivot and the center of mass, M is the mass of the object and I is the moment of inertia of the object *about the given axis*. (Note: the axis is probably *not* at the center of mass; if it were, the mass wouldn't oscillate!)

Following the usual procedure we find the period T:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgd}} \tag{4.22}$$

4.2 Worked Examples

4.2.1 Simple Harmonic Motion

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1. The displacement of a particle at $t = 0.25\,\mathrm{s}$ is given by the expression $x = (4.0\,\mathrm{m})\cos(3.0\pi t + \pi)$ where x is in meters and t is in seconds. Determine (a) the frequency and period of the motion, (b) the amplitude of the motion, (c) the phase constant, and (d) the displacement of the particle at $t = 0.25\,\mathrm{s}$. [Ser4 13-1]

(a) We compare the given function $x(t)$ with the standard form for simple harmonic motion given in Eq. 4.1. This gives us the angular frequency ω:

$$\omega = 3.0\pi\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

and from this we can get the frequency and period:

$$f = \frac{\omega}{2\pi} = \frac{3.0\pi\,\frac{\mathrm{rad}}{\mathrm{s}}}{2\pi} = 1.50\,\mathrm{Hz}$$

$$T = \frac{1}{f} = \frac{1}{(1.50\,\mathrm{s}^{-1})} = 0.667\,\mathrm{s}$$

(b) We easily read off the amplitude as the factor (a length) which multiplies the cosine function:

$$A = 4.0\,\mathrm{m}$$

(c) Again, comparison with Eq. 4.1 gives

$$\phi = \pi$$

(d) At $t = 0.25\,\mathrm{s}$ the displacement (i.e. the coordinate) of the particle is:

$$\begin{aligned} x(0.25\,\mathrm{s}) &= (4.0\,\mathrm{m})\cos((3.0\pi)(0.25) + \pi) = (4.0\,\mathrm{m})\cos((1.75)\pi) \\ &= (4.0\,\mathrm{m})(0.707) = 2.83\,\mathrm{m} \end{aligned}$$

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2. A loudspeaker produces a musical sound by means of the oscillation of a diaphragm. If the amplitude of oscillation is limited to $1.0 \times 10^{-3}\,\mathrm{mm}$, what frequencies will result in the magnitude of the diaphragm's acceleration exceeding g? [HRW5 16-5]

We are given the amplitude of the diaphragm's motion, $A = 1.0\times10^{-3}\,\mathrm{mm} = 1.0\times10^{-6}\,\mathrm{m}$. From Eq. 4.6, the maximum value of the acceleration is $a_{\max} = A\omega^2$. So then the angular frequency that results in a maximum acceleration of g is

$$\omega^2 = \frac{a_{\max}}{A} = \frac{(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(1.0\times10^{-6}\,\mathrm{m})} = 9.8\times10^{6}\,\mathrm{s}^{-2}$$

$$\Longrightarrow \qquad \omega = 3.1 \times 10^3\,\mathrm{s}^{-1} \ .$$

This corresponds to a *frequency* of

$$f = \frac{\omega}{2\pi} = \frac{(3.1 \times 10^3\,\mathrm{s}^{-1})}{2\pi} = 5.0 \times 10^2\,\mathrm{Hz}$$

At frequencies larger than 500 Hz, the acceleration of the diaphragm will exceed g.

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3. The scale of a spring balance that reads from 0 to 15.0 kg is 12.0 cm long. A package suspended from the balance is found to oscillate vertically with a frequency of 2.00 Hz. (a) What is the spring constant? (b) How much does the package weigh? [HRW5 16-6]

(a) The data in the problem tells us us that the spring within the balance *increases in length* by 12.0 cm when a *weight* of

$$W = mg = (15.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 147\,\mathrm{N}$$

is pulls downward on its end. So the force constant of the spring must be

$$k = \frac{F}{x} = \frac{(147\,\mathrm{F})}{(12 \times 10^{-2}\,\mathrm{m})} = 1225\,\tfrac{\mathrm{N}}{\mathrm{m}}$$

(b) Eq. 4.11 we have the frequency of oscillation of the mass–spring system in terms of the spring constant and the attached mass. We have the frequency and spring constant and we can solve to get the mass of the package:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \Longrightarrow \qquad m = \frac{k}{4\pi^2 f^2}$$

Plug in the numbers:

$$m = \frac{(1225\,\tfrac{\mathrm{N}}{\mathrm{m}})}{4\pi^2(2.00\,\mathrm{s}^{-1})^2} = 7.76\,\mathrm{kg}$$

That's the *mass* of the package; its *weight* is

$$W = mg = (7.76\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 76\,\mathrm{N}$$

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4. In an electric shaver, the blade moves back and forth over a distance of 2.0 mm in simple harmonic motion, with frequency 120 Hz. Find (a) the amplitude, (b) the maximum blade speed, and (c) the magnitude of the maximum acceleration. [HRW5 16-9]

(a) The problem states that the *full* distance that the blade travels on each back-and-forth swing is 2.0 mm, but the full swing distance is *twice* the amplitude. So the amplitude of the motion is $A = 1.0\,\mathrm{mm}$.

(b) From Eq. 4.6 we have the maximum speed of an oscillating mass in terms of the amplitude and frequency:

$$v_{\rm max} = \omega A = 2\pi f A = 2\pi(120\,{\rm s}^{-1})(1.0 \times 10^{-3}\,{\rm m}) = 0.75\,\tfrac{\rm m}{\rm s}$$

(c) From Eq. 4.6 we also have magnitude of the maximum acceleration of an oscillating mass in terms of the amplitude and frequency:

$$a_{\rm max} = \omega^2 A = (2\pi f)^2 A = 4\pi^2(120\,{\rm s}^{-1})^2(1.0 \times 10^{-3}\,{\rm m}) = 570\,\tfrac{\rm m}{\rm s^2}$$

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5. The end of one of the prongs of a tuning fork that executes simple harmonic motion of frequency 1000 Hz has an amplitude of 0.40 mm. Find (a) the maximum acceleration and (b) the maximum speed of the end of the prong. Find (c) the acceleration and (d) the speed of the end of the prong when the end has a displacement of 0.20 mm [HWR5 16-22]

(a) Since we have the amplitude A of the prong's motion, and we can easily find the angular frequency ω:

$$\omega = 2\pi f = 2\pi(1000\,{\rm Hz}) = 6.28 \times 10^3\,{\rm s}^{-1}$$

we can use Eq. 4.6 to find the maximum value of a:

$$\begin{aligned} a_{\rm max} &= \omega^2 A = (6.28 \times 10^3\,{\rm s}^{-1})^2(0.400 \times 10^{-3}\,{\rm m}) \\ &= 1.6 \times 10^4\,\tfrac{\rm m}{\rm s^2} \end{aligned}$$

(b) Likewise, from the same equation we find the maximum speed of the prong's tip:

$$\begin{aligned} v_{\rm max} &= \omega A = (6.28 \times 10^3\,{\rm s}^{-1})(0.400 \times 10^{-3}\,{\rm m}) \\ &= 2.5\,\tfrac{\rm m}{\rm s} \end{aligned}$$

(c) Equation 4.7 relates the acceleration a and coordinate x at *all* times. When the displacement of the prong's tip is 0.20 mm (half of its maximum) we find

$$a = -\omega^2 x = -(6.28 \times 10^3\,{\rm s}^{-1})^2(0.20 \times 10^{-3}\,{\rm m}) = -7.9 \times 10^3\,\tfrac{\rm m}{\rm s^2}$$

(d) We have already given a relation between $|v|$ (speed) and x in Eq. 4.8. We use it here to find the speed when $x = 0.20$ mm:

$$\begin{aligned} |v| &= \omega A\sqrt{1 - \left(\frac{x(t)}{A}\right)^2} \\ &= (6.28 \times 10^3\,{\rm s}^{-1})(0.40 \times 10^{-3}\,{\rm m})\sqrt{1 - \left(\frac{0.20\,{\rm mm}}{0.40\,{\rm mm}}\right)^2} \\ &= 2.2\,\tfrac{\rm m}{\rm s} \end{aligned}$$

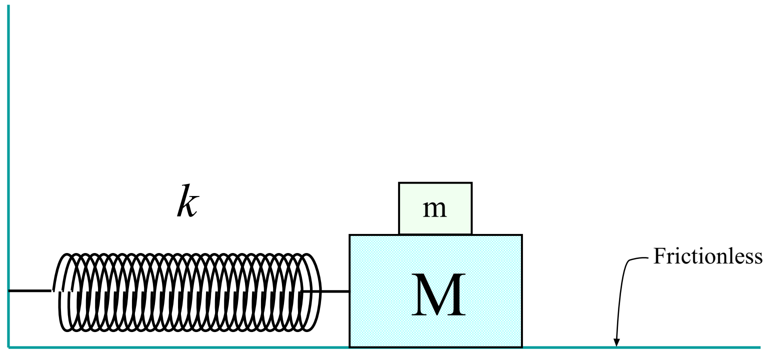

Figure 4.6: Mass M is attached to a spring and oscillates on a frictionless surface. Another block of mass m is on top!

4.2.2 Mass Attached to a Spring

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6. A $7.00-\text{kg}$ **mass is hung from the bottom end of a vertical spring fastened to an overhead beam. The mass is set into vertical oscillations having a period of** $2.60\,\text{s}$**. Find the force constant of the spring.** [Ser4 13-11]

The formulae in Eq. 4.11 hold even if the mass–spring system oscillates vertically (just as long as we can neglect the mass of the spring). Then we can solve for the force constant:

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad \Longrightarrow \qquad T^2 = \frac{4\pi^2 m}{k} \qquad \Longrightarrow \qquad k = \frac{4\pi^2 m}{T^2}$$

and the numbers give us

$$k = \frac{4\pi^2(7.00\,\text{kg})}{(2.60\,\text{s})^2} = 40.9\,\tfrac{\text{kg}}{\text{s}^2} = 40.9\,\tfrac{\text{N}}{\text{m}} \ .$$

The force constant of the spring is $40.9\,\frac{\text{N}}{\text{m}}$.

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7. Two blocks ($m = 1.0\,\text{kg}$ **and** $M = 10\,\text{kg}$**) and a spring (**$k = 200\,\frac{\text{N}}{\text{m}}$**) are arranged on a horizontal, frictionless surface as shown in Fig. 4.6. The coefficient of static friction between the two blocks is** 0.40**. What is the maximum possible amplitude of simple harmonic motion of the spring–block system if no slippage is to occur between the blocks?** [HRW5 16-25]

We first look at what happens when the two blocks oscillate together. In that case it is legal to regard the mass on the spring as a *single* mass whose value is $\mathcal{M} = M + m = 11.0\,\text{kg}$. We know the spring constant, so using Eq. 4.10 the angular frequency of the motion is

$$\omega = \sqrt{\frac{k}{\mathcal{M}}} = \sqrt{\frac{200\,\frac{\text{N}}{\text{m}}}{11.0\,\text{kg}}} = 4.26\,\text{s}^{-1}$$

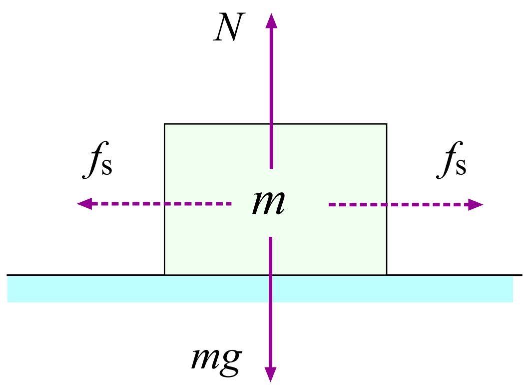

Figure 4.7: The forces acting on mass m in Example 7. (The force of static friction changes direction and magnitude during the motion of mass m.)

During the motion, the large mass oscillates with this frequency and so does the small mass since they move together. But note, the spring is attached only to the *large* mass; what is making the *small* mass move back and forth? The answer is static friction.

We make a diagram of the forces which act on the small mass. This is shown in Fig. 4.7. We have the force of gravity mg pointing down, the normal force N from the big block pointing up and also the force of static friction $\mathbf{f}_s$, which can point either to the right or to the left, depending on the current position of m during the oscillation! The magnitude and direction of the static friction force $\mathbf{f}_s$ are not constant; the value of $\mathbf{f}_s$ depends on the acceleration of the co-moving blocks (assuming there is no slipping so that they are indeed co-moving).

There is no vertical motion of the small block so clearly

$$N = mg = (1.00\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 9.80\,\mathrm{N} \ .$$

But having the normal force (between the surfaces of the two blocks) we know the maximum possible magnitude of the static friction force, namely:

$$f_s^{\max} = \mu_s F_N = \mu_s mg$$

and since that is the only sideways force on mass m, from Newton's 2nd Law, the maximum possible magnitude of its acceleration — assuming no slipping — is

$$a_{\max}^{\mathrm{no-slip}} = \frac{f_s^{\max}}{m} = \frac{\mu_s mg}{m} = \mu_s g \ .$$

Now, if the two blocks are moving together and oscillating with amplitude A, then the maximum value of the acceleration is given by Eq. 4.6, namely $a_{\max} = \omega^2 A$, which of course will get larger if A gets larger. By equating this maximum acceleration of the motion to the value we just found, we arrive at a condition on the maximum amplitude A such that no slipping will occur:

$$a_{\max}^{\mathrm{no-slip}} = \omega^2 A_{\max} \qquad \Longrightarrow \qquad \mu_s g = \omega^2 A_{\max}$$

which gives:

$$A_{\rm max} = \frac{\mu_{\rm s} g}{\omega^2} = \frac{(0.40)(9.80\,\frac{\rm m}{\rm s^2})}{(4.26\,{\rm s}^{-1})^2} = 0.216\,{\rm m}$$

4.2.3 Energy and the Simple Harmonic Oscillator

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8. A particle executes simple harmonic motion with an amplitude of $3.00\,{\rm cm}$**. At what displacement from the midpoint of its motion does its speed equal one half of its maximum speed?** [Ser4 13-23]

The maximum speed occurs in the center of the motion, where there is no potential energy. So the total energy is given by

$$E = \tfrac{1}{2} m v_{\rm max}^2$$

At the point(s) where $v = \frac{1}{2} v_{\rm max}$ the potential energy is not zero; rather it is given by

$$\begin{aligned} U &= E - \tfrac{1}{2} m v^2 \\ &= \tfrac{1}{2} m v_{\rm max}^2 - \tfrac{1}{2} m v^2 \\ &= \tfrac{1}{2} m v_{\rm max}^2 - \tfrac{1}{2} m \left(\frac{v_{\rm max}}{2}\right)^2 \\ &= \left(\frac{1}{2} - \frac{1}{8}\right) m v_{\rm max}^2 = \tfrac{3}{8} m v_{\rm max}^2 \end{aligned}$$

But we also have from Eq. 4.16 the result

$$U(x) = \tfrac{1}{2} \frac{m v_{\rm max}^2}{A^2} x^2$$

And combining these expressions gives the corresponding value of x:

$$\tfrac{1}{2} \frac{m v_{\rm max}^2}{A^2} x^2 = \tfrac{3}{8} m v_{\rm max}^2$$

Solve for x:

$$x^2 = \frac{3}{4} A^2 \qquad \Longrightarrow \qquad x = \pm \frac{\sqrt{3}}{2} A = \pm \frac{\sqrt{3}}{2} (3.00\,{\rm cm}) = \pm 2.60\,{\rm cm}$$

The mass has half its maximum speed at $x = \pm 2.60\,{\rm cm}$.

The problem can also be worked just using Eqs. 4.1 and 4.4. The problem gives no data about any specific value of t so we are free to choose $\phi = 0$ for simplicity. Then

$$x(t) = A\cos(\omega t) \qquad \text{and} \qquad v(t) = -\omega A \sin(\omega t) = -v_{\rm max}\sin(\omega t)$$

and for the times t at which the speed of the mass is half the maximum value, we must have the condition

$$\sin(\omega t) = \pm\tfrac{1}{2} \ .$$

But when this is true we have

$$\cos^2(\omega t) = 1 - \sin^2(\omega t) = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

or

$$\cos(\omega t) = \pm\frac{\sqrt{3}}{2}$$

and that gives

$$x = \pm A\frac{\sqrt{3}}{2} = \pm(3.00\,\mathrm{cm})\frac{\sqrt{3}}{2} = \pm 2.60\,\mathrm{cm}$$

4.2.4 The Simple Pendulum

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9. A simple pendulum has a period of $2.50\,\mathrm{s}$. (a) What is its length? (b) What would its period be on the Moon, where $g_{\mathrm{Moon}} = 1.67\,\frac{\mathrm{m}}{\mathrm{s}^2}$? [Ser4 13-25]

(a) Using Eq. 4.19 we solve for the length:

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \Longrightarrow \qquad T^2 = 4\pi^2\frac{L}{g} \qquad \Longrightarrow \qquad L = \frac{T^2 g}{4\pi^2}$$

and the numbers give:

$$L = \frac{(2.50\,\mathrm{s})^2(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{4\pi^2} = 1.55\,\mathrm{m}$$

The length of the pendulum is $1.55\,\mathrm{m}$.

(b) If we take this pendulum to the Moon, its length will be the same, but the acceleration of gravity will be different. Using the new value of g in Eq. 4.19 we find

$$T_{\mathrm{Moon}} = 2\pi\sqrt{\frac{L}{g_{\mathrm{Moon}}}} = 2\pi\sqrt{\frac{(1.55\,\mathrm{m})}{(1.67\,\frac{\mathrm{m}}{\mathrm{s}^2})}} = 6.06\,\mathrm{s}$$

The pendulum's period on the Moon is $6.06\,\mathrm{s}$.

---

10. If a simple pendulum with length $1.50\,\mathrm{m}$ makes 72.0 oscillations in $180\,\mathrm{s}$, what is the acceleration of gravity at its location? [HRW5 16-59]

We find the frequency f of this pendulum:

$$f = \frac{72.0}{180\,\mathrm{s}} = 0.400\,\mathrm{Hz}$$

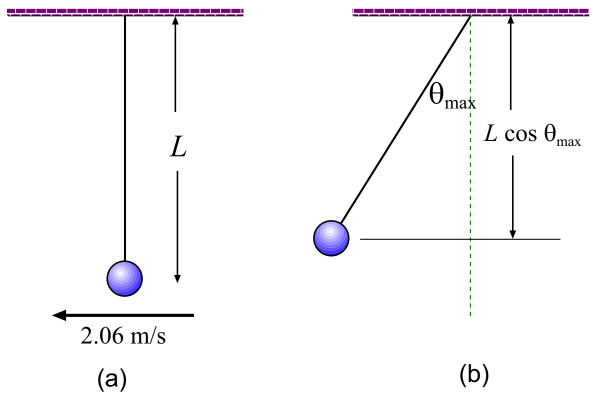

Figure 4.8: (a) Pendulum starts with speed $2.06\,\frac{\mathrm{m}}{\mathrm{s}}$ at the bottom of the swing. (b) It attains a maximum angular displacement $\theta_{\rm max}$.

Then from Eq. 4.19 we can solve for the value of g:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad \Longrightarrow \qquad (2\pi f)^2 = \frac{g}{L} \qquad \Longrightarrow \qquad g = 4\pi^2 f^2 L$$

Plug in the numbers:

$$g = 4\pi^2 (0.400\,\mathrm{s}^{-1})^2 (1.50\,\mathrm{m}) = 9.47\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration of gravity at this location is $9.47\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

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11. A simple pendulum having a length of $2.23\,\mathrm{m}$ and a mass of $6.74\,\mathrm{kg}$ is given an initial speed of $2.06\,\frac{\mathrm{m}}{\mathrm{s}}$ at its equilibrium position. Assume it undergoes simple harmonic motion and determine its (a) period, (b) total energy, and (c) maximum angular displacement. [Ser4 13-59]

The problem is diagrammed in Fig. 4.8 (a).

I will answer the parts of this question in a different order; one reason for this is that part (c) (maximum value of θ) can clearly be found using energy conservation. Finding the maximum angular displacement will then give the period.

First off, if we measure height from the bottom of the pendulum's swing, then in its initial position it has no potential energy but a kinetic energy equal to

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(6.74\,\mathrm{kg})(2.06\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 14.3\,\mathrm{J}$$

so the total energy of the system is $14.3\,\mathrm{J}$.

Now when the mass reaches its maximum angular displacement (say, $\theta_{\rm max}$) it is at a height

$$y_{\rm max} = L - L\cos\theta_{\rm max} = L(1-\cos\theta_{\rm max}) \ .$$

At that time *all* of the energy of the particle is potential energy and using energy conservation, we can solve for $\theta_{\rm max}$:

$$E = mgy_{\rm max} = mgL(1-\cos\theta_{\rm max}) = 14.3\,\mathrm{J}$$

$$(1-\cos\theta_{\max}) = \frac{(14.3\,\mathrm{J})}{mgL} = \frac{(14.3\,\mathrm{J})}{(6.74\,\mathrm{kg}(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})(2.23\,\mathrm{m})} = 9.71\times10^{-2}$$

$$\cos\theta_{\max} = 1 - 9.71\times10^{-2} = 9.03\times10^{-1}$$

$$\theta_{\max} = 25.4^\circ = 0.444\,\mathrm{rad}$$

This is the *exact* answer for $\theta_{\max}$. Now, one might wonder if 25.4° is small enough so that our calculation of the period of the motion is very accurate, but we forge on anyway!

Now, we are given the linear speed at the bottom of the swing, but the pendulum's (harmonic) motion has to do with its *angle*. We need to relate the two.

From Eq. 1.10 we can get the *angular velocity* of the mass at the bottom of the swing:

$$\left(\frac{d\theta}{dt}\right)_0 = \frac{v_0}{L} = \frac{(2.06\,\frac{\mathrm{m}}{\mathrm{s}})}{(2.23\,\mathrm{m})} = 0.924\,\frac{\mathrm{rad}}{\mathrm{s}}$$

But from 4.20 we have $\theta(t) = \theta_{\max}\cos(\omega t+\phi)$ so that the angular velocity of the pendulum at all times is

$$\frac{d\theta}{dt} = -\omega\theta_{\max}\cos(\omega t+\phi)$$

so that the maximum angular speed (namely at the bottom of the swing) is

$$\left(\frac{d\theta}{dt}\right)_{\max} = \omega\theta_{\max}$$

and this is the same as the $0.924\,\frac{\mathrm{rad}}{\mathrm{s}}$ found above. So we can get ω:

$$\left(\frac{d\theta}{dt}\right)_{\max} = 0.924\,\frac{\mathrm{rad}}{\mathrm{s}} = \omega\theta_{\max} \qquad \Longrightarrow \qquad \omega = \frac{(0.924\,\frac{\mathrm{rad}}{\mathrm{s}})}{(0.444\,\mathrm{rad})} = 2.08\,\frac{\mathrm{rad}}{\mathrm{s}}$$

(It is true that the units don't look right on that last one, but keep in mind that "radian" is really dimensionless.)

Having ω, the angular frequency of the pendulum's *oscillations*, we go on to get the period:

$$T = \frac{2\pi}{\omega} = 3.02\,\mathrm{s}$$

Summing up what problem asked for, we have:

$$\text{(a)}\quad T = 3.02\,\mathrm{s} \qquad \text{(b)}\quad E = 14.3\,\mathrm{J} \qquad \text{(c)}\quad \theta_{\max} = 0.444\,\mathrm{rad}$$

4.2.5 Physical Pendulums

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12. A physical pendulum in the form of a planar body moves in simple harmonic motion with a frequency of $0.450\,\mathrm{Hz}$. If the pendulum has a mass of $2.20\,\mathrm{kg}$ and the pivot is located $0.350\,\mathrm{m}$ from the center of mass, determine the moment of inertia of the pendulum. [Ser4 13-33]

From Eq. 4.22 we have an expression for the frequency of a "physical pendulum":

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{Mgd}{I}}$$

We have all the values that we need to solve for I:

$$f^2 = \frac{Mgd}{4\pi^2 I} \qquad \Longrightarrow \qquad I = \frac{Mgd}{4\pi^2 f^2}$$

Plug in the numbers:

$$I = \frac{(2.20\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})(0.350\,\mathrm{m})}{4\pi^2(0.450\,\mathrm{s}^{-1})^2} = 0.944\,\mathrm{kg}\cdot\mathrm{m}^2$$

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13. A thin disk of mass $5\,\mathrm{kg}$ and radius $20\,\mathrm{cm}$ is suspended by a horizontal axis perpendicular to the disk through the rim. The disk is displaced slightly from equilibrium and released. Find the period of the subsequent simple harmonic motion. [Tip4 14-57]

We need to find the moment of inertia of the disk when it rotates around an axis *at the rim* of the disk. For this, we use the Parallel Axis Theorem of Chapter 1. With I_{CM} given by $I_{\mathrm{CM}} = \frac{1}{2}MR^2$ and recognizing that in this problem, the axis has been shifted a distance $D = R$ away from the center of mass, we find:

$$I = I_{\mathrm{CM}} + MD^2 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

Then we can find the period of the oscillatory motion from Eq. 4.22. Note, the distance of the axis from the center of mass (called D in that formula) is R:

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{\frac{3}{2}MR^2}{MgR}} = 2\pi\sqrt{\frac{3R}{2g}} = 2\pi\sqrt{\frac{3(0.20\,\mathrm{m})}{2(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}} = 1.1\,\mathrm{s}$$

The period of this pendulum is $1.1\,\mathrm{s}$. Interestingly enough, the answer did not depend on the mass of the disk.