Chapter 2

Rolling Motion; Angular Momentum

2.1 The Important Stuff

2.1.1 Rolling Without Slipping

When a round, symmetric rigid body (like a uniform cylinder or sphere) of radius R rolls without slipping on a horizontal surface, the distance though which its center travels (when the wheel turns by an angle θ) is the same as the arc length through which a point on the edge moves:

$$\Delta x_{\mathrm{CM}} = s = R\theta \tag{2.1}$$

These quantities are illustrated in Fig. 2.1.

The speed of the center of mass of the rolling object, $v_{CM} = \frac{dx_{CM}}{dt}$ and its angular speed are related by

$$v_{\mathrm{CM}} = R\omega \tag{2.2}$$

and the acceleration magnitude of the center of mass is related to the angular acceleration by:

$$a_{\mathrm{CM}} = R\alpha \tag{2.3}$$

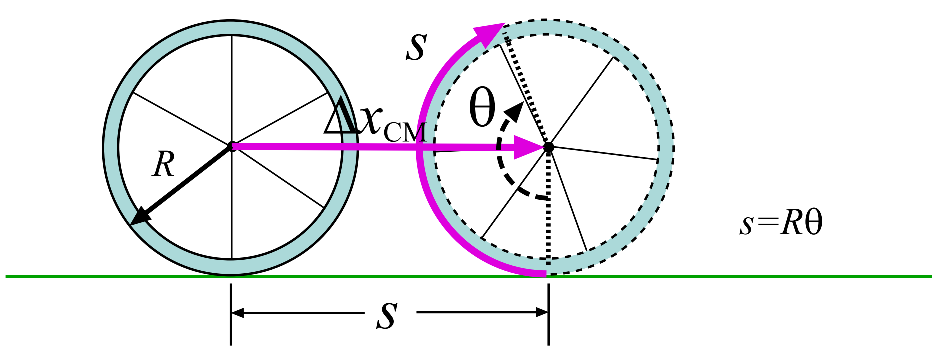

Figure 2.1: Illustration of the relation between Δx, s, R and θ for a rolling object.

The kinetic energy of the object is:

$$K_{\rm roll} = \tfrac{1}{2}I_{\rm CM}\omega^2 + \tfrac{1}{2}Mv_{\rm CM}^2 \ . \tag{2.4}$$

The first term on the right side represents the rotational kinetic energy of the object about its symmetry axis; the second term represents the kinetic energy the object would have if it moved along with speed $v_{\rm CM}$ without rotating (i.e. just translational motion). We can remember this relation simply as: $K_{\rm roll} = K_{\rm rot} + K_{\rm trans}$.

When a wheel rolls without slipping there *may* be a frictional force of the surface on the wheel. If so, it is a force of *static* friction (which does no work) and depending on the situation it could point in the same direction or opposite the motion of the center of mass; in all cases it tends to oppose the tendency of the wheel to slide.

2.1.2 Torque as a Vector (A Cross Product)

In the last chapter we gave a definition for the torque τ acting on a rigid body rotating around a fixed axis. We now give a more general definition for "torque"; we define the torque acting on a single particle (relative to some fixed point O) when a force acts on it.

Suppose the (instantaneous) position vector of a particle (relative to the origin O) is $\mathbf{r}$ and a single force $\mathbf{F}$ acts on it. Then the torque $\boldsymbol{\tau}$ acting on the particle is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{2.5}$$

If ϕ is the angle between the position vector $\mathbf{r}$ and the force $\mathbf{F}$ then the torque $\boldsymbol{\tau}$ has magnitude

$$\tau = rF\sin\phi$$

2.1.3 Angular Momentum of a Particle and of Systems of Particles

There is yet more important quantity having to do with rotations that will be of help in solving problems involving rotating objects; just as the linear momentum $\mathbf{p}$ was of importance in problems with interacting particles, the *angular momentum* of objects which have motion about a given axis will be useful when these objects interact with one another. Admittedly, some of the first definitions and theorems will be rather abstract! But we will soon apply the ideas to simple objects which rotate around an axis and then the theorems and examples will be quite down–to–earth.

We start with a fundamental definition; if a particle has position vector $\mathbf{r}$ and linear momentum $\mathbf{p}$, both relative to some origin O, then the **angular momentum** of that particle (relative to the origin) is defined by:

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) \tag{2.6}$$

Angular momentum has units of $\frac{\rm kg\cdot m^2}{\rm s}$.

One can show that the net torque on a particle is equal to the time derivative of its angular momentum:

$$\sum \boldsymbol{\tau} = \frac{d\boldsymbol{\ell}}{dt} \tag{2.7}$$

This relation is analogous to the relation $\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$ from linear motion.

For a set of mass points in motion, we define the total angular momentum as the (vector) sum of the individual angular momenta:

$$\mathbf{L} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 + \boldsymbol{\ell}_3 + \ldots$$

When we consider the *total* angular momentum, we can prove a theorem which is a bit different in its content than Eq. 2.7. It's a bit subtle; when the particles in a system all move around they will be acted upon by forces from *outside* the system but also by the forces they all exert on one another. What the theorem says is that the rate of change of the *total* angular momentum just comes from the torques arising from forces exerted from outside the system. This is useful because the external torques aren't so hard to calculate.

The theorem is:

$$\sum \boldsymbol{\tau}_{\rm ext} = \frac{d\mathbf{L}}{dt} \tag{2.8}$$

This tells us that when the sum of external torques is zero then **L** is constant (conserved). We will encounter this theorem most often in problems where there is rotation about a fixed axis (and then once again we will only deal with the z components of $\boldsymbol{\tau}$ and $\mathbf{L}$).

2.1.4 Angular Momentum for Rotation About a Fixed Axis

An extended object is really a set of mass points, and it has a total angular momentum (vector) about a given origin. We will keep things simple by considering only rotations about an axis which is fixed in direction (say, the z direction), and for that case we only need to consider the component of **L** which lies along this axis, L_z. So, for rotation about a fixed axis the "angular momentum" of the rigid object is (for our purposes) just a *number*, L. Furthermore, one can show that if the angular velocity of the object is ω and its moment of inertia about the given axis is I, then its angular momentum about the axis is

$$L = I\omega \tag{2.9}$$

Again, there is a correspondence with the equations for linear motion:

$$p_x = mv_x \qquad \Leftrightarrow \qquad L = I\omega$$

2.1.5 The Conservation of Angular Momentum

In the chapter on Momentum (in Vol. 1) we used an important fact about systems for which there is no (net) external *force* acting: The total momentum remains the same. One can show a similar theorem which concerns *net external torques* and *angular* momenta.

For a system on which there is no net external torque, the total angular momentum remains constant: $L_i = L_f$. This principle is known as the **Conservation of Angular Momentum**.

2.2 Worked Examples

2.2.1 Rolling Without Slipping

1. An automobile traveling $80.0\,\mathrm{km/hr}$ **has tires of** $75.0\,\mathrm{cm}$ **diameter. (a) What is the angular speed of the tires about the axle? (b) If the car is brought to a stop uniformly in** 30.0 **turns of the tires (without skidding), what is the angular acceleration of the wheels? (c) How far does the car move during the braking?** [HRW5 12-3]

(a) We know that the speed of the center of mass of *each wheel* is $80.0\,\mathrm{km/hr}$. And the radius of each wheel is $R = (75.0\,\mathrm{cm})/2 = 37.5\,\mathrm{cm}$. Converting the speed to $\frac{\mathrm{m}}{\mathrm{s}}$, we have:

$$80\,\tfrac{\mathrm{km}}{\mathrm{h}} = \left(80\,\tfrac{\mathrm{km}}{\mathrm{h}}\right)\left(\frac{1\,\mathrm{h}}{3600\,\mathrm{s}}\right)\left(\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}}\right) = 22.2\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

From the relation between v_{CM} and ω for an object which rolls without slipping, we have:

$$v_{\mathrm{CM}} = \omega R \qquad \Longrightarrow \qquad \omega = \frac{v_{\mathrm{CM}}}{R}$$

and we get

$$\omega = \frac{(22.2\,\frac{\mathrm{m}}{\mathrm{s}})}{(0.375\,\mathrm{m})} = 59.3\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

The angular speed of the wheel is $59.3\,\frac{\mathrm{rad}}{\mathrm{s}}$.

(b) As the car comes to a halt, the tires go through 30.0 turns. Thus they have an *angular* displacement of (with $\theta_0 = 0$):

$$\theta = (30.0\,\mathrm{rev})\left(\frac{2\pi\,\mathrm{rad}}{1\,\mathrm{rev}}\right) = 188.5\,\mathrm{rad} \quad .$$

Also, when the wheel has come to a halt, its angular velocity is zero!

So we have the initial and final angular velocities and the angular displacement. We can get the angular acceleration of the wheel from Eq. 1.8. From that equation we get:

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta} = \frac{(0\,\frac{\mathrm{rad}}{\mathrm{s}})^2 - (59.3\,\frac{\mathrm{rad}}{\mathrm{s}})^2}{2(188.5\,\mathrm{rad})} = -9.33\,\tfrac{\mathrm{rad}}{\mathrm{s}^2}$$

The magnitude of the wheels' angular acceleration is $9.33\,\frac{\mathrm{rad}}{\mathrm{s}^2}$. The minus sign in our result indicates that α goes in the sense *opposite* to that of the initial angular velocity (and angular displacement) of the wheel during the stopping.

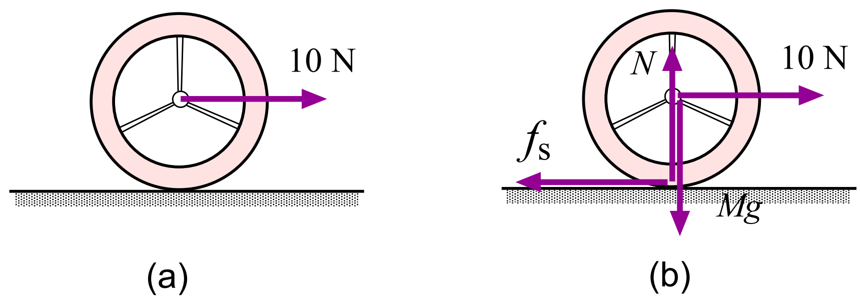

Figure 2.2: (a) Constant horizontal applied to a rolling wheel in Example 3. (b) The forces acting on the wheel, with the points of application as indicated.

(c) As we saw, the *angular* displacement of any wheel during the stopping was 188.5 rad. The radius of the wheel is $R = 0.375\,\mathrm{m}$, so from Eq. 2.1 the linear displacement of the wheel (i.e. its center) is:

$$x_{\mathrm{CM}} = R\theta = (0.375\,\mathrm{m})(188.5\,\mathrm{rad}) = 70.7\,\mathrm{m}$$

so the car goes 70.7 m before coming to a halt.

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2. A bowling ball has a mass of $4.0\,\mathrm{kg}$, a moment of inertia of $1.6 \times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2$ and a radius of $0.10\,\mathrm{m}$. If it rolls down the lane without slipping at a linear speed of $4.0\,\frac{\mathrm{m}}{\mathrm{s}}$, what is its total energy? [Ser4 11-5]

The total (kinetic) energy of an object which rolls without slipping is given by Eq. 2.4. To use this equation we have everything we need except the angular speed of the ball. From Eq. 2.2 it is related to the linear velocity of the ball by $v_{\mathrm{CM}} = R\omega$, so the angular speed is

$$\omega = \frac{v_{\mathrm{CM}}}{R} = \frac{(4.0\,\frac{\mathrm{m}}{\mathrm{s}})}{(0.10\,\mathrm{m})} = 40.0\,\frac{\mathrm{rad}}{\mathrm{s}}$$

and then the kinetic energy is

$$\begin{aligned} K_{\mathrm{roll}} &= \tfrac{1}{2} I_{\mathrm{CM}}\omega^2 + \tfrac{1}{2} M v_{\mathrm{CM}}^2 \\ &= \tfrac{1}{2}(1.6 \times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2)(40.0\,\tfrac{\mathrm{rad}}{\mathrm{s}})^2 + \tfrac{1}{2}(4.0\,\mathrm{kg})(4.0\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 \\ &= 44.8\,\mathrm{J} \end{aligned}$$

The total kinetic energy of the ball is 44.8 J.

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3. A constant horizontal force of $10\,\mathrm{N}$ is applied to a wheel of mass $10\,\mathrm{kg}$ and radius $0.30\,\mathrm{m}$ as shown in Fig. 2.2. The wheel rolls without slipping on the horizontal surface, and the acceleration of its center of mass is $0.60\,\frac{\mathrm{m}}{\mathrm{s}^2}$. (a) What are the magnitude and direction of the frictional force on the wheel? (b) What is the rotational inertia of the wheel about an axis through its center of mass and perpendicular to the plane of the wheel? [HRW5 12-9]

(a) The forces which act on the wheel along with where these forces are applied are shown in Fig 2.2 (b). In addition to the applied force of 10 N which points to the right, there is a force of static friction between the surface and the wheel (of magnitude f_s), which for now we *draw* pointing to the *left* (we can ask: Does it really point that way?). There are vertical forces acting on the wheel (from gravity and the normal force of the surface) but these clearly cancel out and for now we don't need to worry about them.

Even though the wheel will be rolling during its motion, Newton's 2$^{\rm nd}$ law still holds, and the sum of the horizontal forces gives ma_x. Here the wheel is clearly accelerating to the *right* and so with the choice of directions given in the figure, we find:

$$\sum F_x = 10.0\,\mathrm{N} - f_s = ma_x = (10\,\mathrm{kg})(0.60\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 6.0\,\mathrm{N}$$

so that

$$f_s = 10.0\,\mathrm{N} - 6.0\,\mathrm{N} = 4.0\,\mathrm{N}$$

and since this is positive, the frictional force does indeed point to the left, as we guessed. Actually, it wasn't so hard to guess that, since only a leftward frictional force could make the wheel rotate clockwise —as we know it must here— but for some problems in rolling motion, the direction of the static friction force may not be so evident.

(b) Rotational inertia is related to net torque and angular acceleration by way of $\tau = I\alpha$. It is true that in this problem the rotating object is also accelerating but it turns out this relation still holds as long as we choose the center of mass to be the rotation axis.

Considering the four forces in Fig. 2.2 (b), the applied force and the (effective) force of gravity are applied at the center of the wheel, so they give *no* torque about its center. The normal force of the surface is applied at the rim, but its direction is *parallel* to the line which joins the application point to the center so it too gives no torque. All that remains is the friction force, applied at a distance R from the center and perpendicular to the line joining this point and the axis; this force gives a clockwise rotation, so if we take the clockwise direction as the positive sense for rotations, then the net torque on the wheel is

$$\tau = +f_s r = (4.0\,\mathrm{N})(0.30\,\mathrm{m}) = 1.2\,\mathrm{N}\cdot\mathrm{m}$$

From Eq. 2.3 we know the angular acceleration of the wheel; it is

$$\alpha = \frac{a_{\rm CM}}{R} = \frac{(0.60\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(0.30\,\mathrm{m})} = 2.0\,\tfrac{\mathrm{rad}}{\mathrm{s}^2}\ .$$

and then from $\tau = I\alpha$ we get the rotational inertia:

$$I = \frac{\tau}{\alpha} = \frac{(1.2\,\mathrm{N}\cdot\mathrm{m})}{(2.0\,\frac{\mathrm{rad}}{\mathrm{s}^2})} = 0.60\,\mathrm{kg}\cdot\mathrm{m}^2$$

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4. A round, symmetrical object of mass M, radius R and moment of inertia I rolls without slipping down a ramp inclined at an angle θ. Find the acceleration of its center of mass.

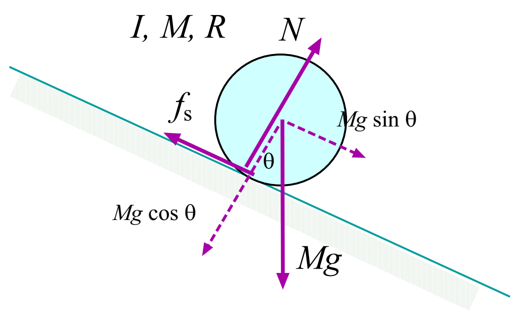

Figure 2.3: Round, symmetrical object with mass M, radius R and moment of inertia I rolls down a ramp sloped at angle θ from the horizontal.

The problem is diagrammed in Fig. 2.3. We show the forces acting on the object and where they are applied. The force of gravity, Mg is (effectively) applied at the center of the object. As usual we decompose this force into its components down the slope and perpendicular to the slope. The slope exerts a normal force N at the point of contact. Finally there is a force of static friction f_s from the surface; this force points *along* the surface and we can pretty quickly see that it must point *up* the slope because it is the friction force which gives the object an angular acceleration, which (here) is in the clockwise sense.

Apply Newton's 2$^{\rm nd}$ law first: The forces perpendicular to the slope cancel, so that $N = Mg\cos\theta$ (but we won't need this fact). If $a_{\rm CM}$ is the acceleration of the CM of the object *down* the slope, then adding the forces *down* the slope gives

$$Mg\sin\theta - f_s = Ma_{\rm CM} \tag{2.10}$$

Now we look at the net *torque* on the rolling object about its center of mass. The force of gravity acts at the center, so it gives no torque. The normal force of the surface acts at the point of contact, but since it is parallel to the line joining the pivot and the point of application, it also gives no torque. The force of friction is applied at a distance R from the pivot and it is perpendicular to the line joining the pivot and point of application. So the friction force gives a torque of magnitude

$$\tau = Rf_s \sin 90^\circ = Rf_s \ .$$

and if we take the *clockwise* sense to be positive for rotations, then the net torque on the object about its CM is

$$\tau_{\rm net} = Rf_s$$

From the relation $\tau = I\alpha$ we then have

$$\tau = Rf_s = I\alpha \tag{2.11}$$

but we can also use the fact that for rolling motion (without slipping) the linear acceleration of the CM and the angular acceleration are related by:

$$a_{\rm CM} = r\alpha \qquad \Longrightarrow \qquad \alpha = \frac{a_{\rm CM}}{R}$$

and using this in Eq. 2.11 gives

$$Rf_s = \frac{Ia_{\rm CM}}{R} \qquad \Longrightarrow \qquad f_s = \frac{Ia_{\rm CM}}{R^2} \tag{2.12}$$

where we choose to isolate f_s (the magnitude of the friction force) so that we can put the result into Eq. 2.10. When we do that, we get:

$$Mg\sin\theta - \frac{Ia_{\rm CM}}{R^2} = Ma_{\rm CM}$$

Now solve for $a_{\rm CM}$:

$$Mg\sin\theta = Ma_{\rm CM} + \frac{Ia_{\rm CM}}{R^2} = \left(M + \frac{I}{R^2}\right)a_{\rm CM}$$

$$a_{\rm CM} = \frac{Mg\sin\theta}{\left(M + \frac{I}{R^2}\right)} \ .$$

If we divide top and bottom of the right side by M, this can be written:

$$a_{\rm CM} = \frac{g\sin\theta}{\left(1 + \frac{I}{MR^2}\right)} \ .$$

Our result is sensible in that if I is very small then $a_{\rm CM}$ is nearly equal to $g\sin\theta$, the result for a mass sliding with no rolling motion.

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5. A uniform sphere rolls down an incline. (a) What must be the incline angle if the linear acceleration of the center of the sphere is to be $0.10\,g$? (b) For this angle, what would be the acceleration of a frictionless block sliding down the incline? [HRW5 12-7]

(a) We will use the formula for $a_{\rm CM}$ (for rolling without slipping down a slope) in a previous problem. Note that we are not given the mass of the sphere! But it turns out that we don't need it, because for a uniform sphere, we have

$$\frac{I}{MR^2} = \frac{2}{5}$$

and as we can see from our earlier result,

$$a_{\rm CM} = \frac{g\sin\theta}{\left(1 + \frac{I}{MR^2}\right)} \ ,$$

$a_{\rm CM}$ just depends on the combination $I/(MR^2)$. Solving this equation for $\sin\theta$ and plugging in the given numbers, we get:

$$\sin\theta = \frac{a_{\rm CM}}{g}\left(1 + \frac{I}{MR^2}\right) = \frac{(0.10g)}{g}(1 + \tfrac{2}{5}) = (0.10)\tfrac{7}{5} = 0.14$$

Which gives us

$$\theta = \sin^{-1}(0.14) = 8.0^\circ$$

(b) As we saw in the Chapter on forces (Volume 1) when a mass slides down a *frictionless* incline its linear acceleration is given by $a = g\sin\theta$. For the slope angle found in part (a), this is

$$a = g\sin\theta = (9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\sin 8.0^\circ = 1.4\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

We can also calculate a/g:

$$\frac{a}{g} = \sin\theta = \sin 8.0^\circ = 0.14$$

so for the frictionless case we have $a = 0.14g$.

2.2.2 Torque as a Vector (A Cross Product)

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6. What are the magnitude and direction of the torque about the origin on a plum (!) located at coordinates $(-2.0\,\mathrm{m}, 0, 4.0\,\mathrm{m})$ due to force F whose only component is (a) $F_x = 6.0\,\mathrm{N}$, (b) $F_x = -6.0\,\mathrm{N}$, (c) $F_z = 6.0\,\mathrm{N}$, and (d) $F_z = -6.0\,\mathrm{N}$? [HRW5 12-21]

(a) The (vector) torque on a point particle is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

I find it easiest to set up the cross product in determinant notation, discussed in Chapter 1 of Volume 1. We note that the units of the result must be $\mathrm{N}\cdot\mathrm{m}$; then the cross product of **r** and **F** is

$$\mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2.0 & 0.0 & 4.0 \\ 6.0 & 0.0 & 0.0 \end{vmatrix}\,\mathrm{N}\cdot\mathrm{m} = (+(4.0)(6.0)\,\mathrm{N}\cdot\mathrm{m})\mathbf{j} = (+24.0\,\mathrm{N}\cdot\mathrm{m})\mathbf{j}$$

The torque $\boldsymbol{\tau}$ has magnitude $24.0\,\mathrm{N}\cdot\mathrm{m}$ and points in the $+y$ direction.

(b) Is is fairly clear that if we had had $F_x = -6.0\,\mathrm{N}$ in part (a), we would have gotten

$$\boldsymbol{\tau} = (-24.0\,\mathrm{N}\cdot\mathrm{m})\mathbf{j}$$

so the torque would have magnitude $24.0\,\mathrm{N}\cdot\mathrm{m}$ and point in the $-y$ direction.

(c) When the only component of **F** is $F_z = 6.0\,\mathrm{N}$, then we have

$$\mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2.0 & 0.0 & 4.0 \\ 0.0 & 0.0 & 6.0 \end{vmatrix}\,\mathrm{N}\cdot\mathrm{m} = (-(-2.0)(6.0)\,\mathrm{N}\cdot\mathrm{m})\mathbf{j} = (+12.0\,\mathrm{N}\cdot\mathrm{m})\mathbf{j}$$

so the torque would have magnitude $12.0\,\mathrm{N}\cdot\mathrm{m}$ and point in the $+y$ direction.

(d) If instead we have only $F_z = -6.0\,\mathrm{N}$ then the sign of the result in part (c) changes, and the torque would have magnitude $24.0\,\mathrm{N}\cdot\mathrm{m}$ and point in the $-y$ direction.

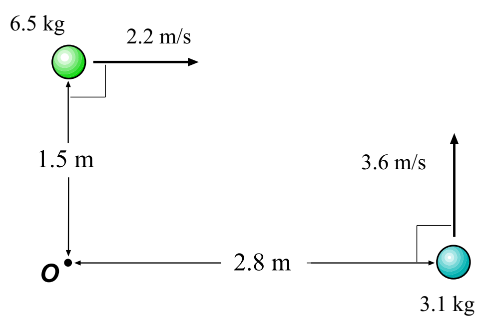

Figure 2.4: Two masses and their motion relative to the point O, as in Example 8.

2.2.3 Angular Momentum of a Particle and of Systems of Particles

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7. The position vector of a particle of mass $2.0\,\mathrm{kg}$ **is given as a function of time by** $\mathbf{r} = (6.0\mathbf{i} + 5.0t\mathbf{j})\,\mathrm{m}$ **when** t **is given in seconds). Determine the angular momentum of the particle as a function of time.** [Ser4 11-17]

The angular momentum of a point mass is given by $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$. The velocity of our particle is given by

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \frac{d}{dt}(6.0\mathbf{i} + 5.0t\mathbf{j})\,\mathrm{m} = (5.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$$

and its momentum is

$$\mathbf{p} = m\mathbf{v} = (2.0\,\mathrm{kg})(5.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}} = (10.0\mathbf{j})\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

Now take the cross product to get $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6.0 & 5.0t & 0.0 \\ 0.0 & 10.0 & 0.0 \end{vmatrix}\,\frac{\mathrm{kg\cdot m^2}}{\mathrm{s}} = (60.0\mathbf{k})\,\frac{\mathrm{kg\cdot m^2}}{\mathrm{s}}$$

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8. Two objects are moving as shown in Fig. 2.4. What is their total angular momentum about point O**?** [HRW5 12-27]

In this problem we must use the definition of the angular momentum of a particle (with respect to some origin O):

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$$

First consider the 3.1 kg mass. Its momentum vector has magnitude

$$p_1 = m_1 v_1 = (3.1\,\mathrm{kg})(3.6\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 11.2\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

and is directed upward in this picture. The vector $\mathbf{r}$ which goes from point O to the particle has magnitude 2.8 m and points to the right. By the right–hand rule for cross products, the vector $\mathbf{r} \times \mathbf{p}$ points up out of the page, which is along the $+z$ axis; and since the two vectors are perpendicular, the magnitude of $\mathbf{r} \times \mathbf{p}$ is

$$|\mathbf{r_1} \times \mathbf{p_1}| = r_1 p_1 \sin 90^\circ = (2.8\,\mathrm{m})(11.2\,\tfrac{\mathrm{kg \cdot m}}{\mathrm{s}}) = 31.2\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s}}$$

and so the angular momentum of *this* particle about O is

$$\boldsymbol{\ell}_1 = (+31.2\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s}})\mathbf{k}$$

The 6.5 kg mass has a momentum vector of magnitude

$$p_2 = m_2 v_2 = (6.5\,\mathrm{kg})(2.2\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 14.3\,\tfrac{\mathrm{kg \cdot m}}{\mathrm{s}}$$

and is directed to the right. The vector $\mathbf{r}$ which goes from the point O to the particle has magnitude 1.5 m and points straight up. By the right–hand rule for cross products, the vector $\mathbf{r} \times \mathbf{p}$ points *into* the page, which is along the $-z$ axis; and since the two vectors are perpendicular, the magnitude of $\mathbf{r} \times \mathbf{p}$ is

$$|\mathbf{r_2} \times \mathbf{p_2}| = r_2 p_2 \sin 90^\circ = (1.5\,\mathrm{m})(14.3\,\tfrac{\mathrm{kg \cdot m}}{\mathrm{s}}) = 21.5\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s}}$$

and so the angular momentum of *this* particle about O is

$$\boldsymbol{\ell}_2 = (-21.5\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s}})\mathbf{k}$$

The total angular momentum of the system is

$$\boldsymbol{\ell} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 = (31.2 - 21.5)\mathbf{k}\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s}} = (+9.8\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s}})\mathbf{k}$$

2.2.4 Angular Momentum for Rotation About a Fixed Axis

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9. A uniform rod rotates in a horizontal plane about a vertical axis through one end. The rod is 6.00 m long, weighs 10.0 N, and rotates at 240 rev/min clockwise when seen from above. Calculate (a) the rotational inertia of the rod about the axis of rotation and (b) the angular momentum of the rod about that axis. [HRW5 12-45]

(a) The *mass* of the rod is

$$M = W/g = (10.0\,\mathrm{N})/(9.80\,\tfrac{\mathrm{m}}{\mathrm{s^2}}) = 10.2\,\mathrm{kg}$$

and its angular velocity in units of $\frac{\mathrm{rad}}{\mathrm{s}}$ is

$$\omega = 240\,\tfrac{\mathrm{rev}}{\mathrm{min}} \left(\frac{2\pi\,\mathrm{rad}}{1\,\mathrm{rev}}\right)\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) = 25.1\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

We know the formula for the moment of inertia of a uniform rod rotating about an axis at one of its ends (see Chapter 1, Fig. 1) so we calculate I as:

$$I = \tfrac{1}{3}ML^2 = \tfrac{1}{3}(10.2\,\mathrm{kg})(6.00\,\mathrm{m})^2 = 12.2\,\mathrm{kg}\cdot\mathrm{m}^2$$

The rotational inertia of the rod (about the given axis) is $12.2\,\mathrm{kg}\cdot\mathrm{m}^2$.

(b) The angular momentum of the rotating rod will be given by $L = I\omega$. We find that the magnitude of the angular momentum is:

$$L = I\omega = (12.2\,\mathrm{kg}\cdot\mathrm{m}^2)(25.1\,\tfrac{\mathrm{rad}}{\mathrm{s}}) = 309\,\tfrac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}}$$

The vector **L** would point upward (along the $+z$ axis if the rotation were *counterclockwise* as seen from above. That is not the case (it is clockwise) so the direction of **L** is downward.

2.2.5 The Conservation of Angular Momentum

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10. Suppose that the Sun runs out of nuclear fuel and suddenly collapses to form a white dwarf star, with a diameter equal to that of the Earth. Assuming no mass loss, what would then be the Sun's new rotation period, which currently is about $25\,\mathrm{days}$**? Assume that the Sun and the white dwarf are uniform, solid spheres; the present radius of the Sun is** $6.96\times10^8\,\mathrm{m}$**.** [HRW5 12-55]

In this simplified account of what will happen to our Sun, its radius will decrease *without any interactions from other masses*, including torques. Without external torques, the angular momentum of the Sun will remain the same, even if it suddenly shrinks to a much smaller size.

The present angular velocity of the Sun (using the given data) is

$$\omega_i = \left(\frac{1\,\mathrm{rev}}{25\,\mathrm{day}}\right)\left(\frac{2\pi\,\mathrm{rad}}{1\,\mathrm{rev}}\right)\left(\frac{1\,\mathrm{day}}{86400\,\mathrm{s}}\right) = 2.91\times10^{-6}\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

and assuming it is a uniform sphere (a bad assumption, actually) its present moment of inertia is

$$I_i = \tfrac{2}{5}MR_i^2$$

(we'll leave it in this form; later on, the value of M will cancel out). Its initial angular momentum is

$$L_i = I_i\omega_i = \tfrac{2}{5}MR_i^2(2.91\times10^{-6}\,\tfrac{\mathrm{rad}}{\mathrm{s}})\ .$$

After the Sun shrinks, it has a new (much smaller) radius, but the *same* mass, M. Its new moment of inertia is

$$I_f = \tfrac{2}{5}MR_f^2$$

and if its final angular velocity is ω_f, then its final angular momentum is

$$L_f = I_f\omega_f = \tfrac{2}{5}MR_f^2\omega_f$$

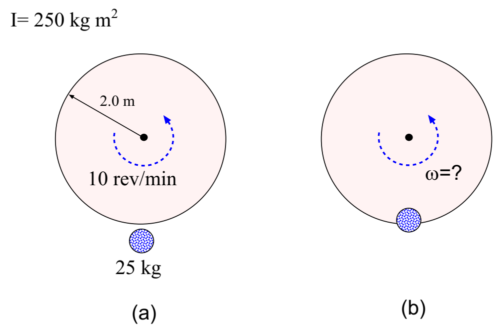

Figure 2.5: Child jumps onto a rotating merry-go-round in Example 11.

Conservation of angular momentum gives us $L_i = L_f$ and so:

$$\tfrac{2}{5}MR_i^2(2.91 \times 10^{-6}\ \tfrac{\text{rad}}{\text{s}}) = \tfrac{2}{5}MR_f^2\omega_f \ .$$

We cancel lotsa things and find:

$$\omega_f = \frac{R_i^2}{R_f^2}(2.91 \times 10^{-6}\ \tfrac{\text{rad}}{\text{s}})$$

The present radius of the Sun is 6.96×10^8 m and the radius of the Earth is 6.37×10^6 m so we find that the angular velocity of the Sun will be

$$\omega_f = \frac{(6.96 \times 10^8\ \text{m})^2}{(6.37 \times 10^6\ \text{m})^2}(2.91 \times 10^{-6}\ \tfrac{\text{rad}}{\text{s}}) = 3.47 \times 10^{-2}\ \tfrac{\text{rad}}{\text{s}} \ .$$

To get the *period* of the Sun's motion, use

$$f = \frac{\omega}{2\pi} \qquad \Longrightarrow \qquad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

So:

$$T_f = \frac{2\pi}{\omega_f} = \frac{2\pi}{(3.47 \times 10^{-2}\ \frac{\text{rad}}{\text{s}})} = 181\,\text{s}\left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 3.0\,\text{min}$$

The squooshed–down Sun will have a rotation period of 3.0 minutes!

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11. A merry-go-round of radius $R = 2.0\,\text{m}$ has a moment of inertia $I = 250\,\text{kg}\cdot\text{m}^2$ and is rotating at $10\,\text{rev/min}$. A $25\,\text{kg}$ child jumps onto the edge of the merry-go-round. What is the new angular speed of the merry-go-round? [Ser4 11-28]

We diagram the problem in Fig. 2.5. In (a) the child is waiting (motionless) by the rim of the rotating wheel and in (b) he/she/it has just stepped on. What do we know about pictures (a) and (b)? We know that if we consider the "system" to be the combination of merry-go-round and child, if the child just steps onto the wheel at its rim there will be *no* *external torques* on this system. And so the total angular momentum of the system will be conserved.

We calculate the total angular momentum that we see in Fig. 2.5 (a). Only the wheel is in motion; its angular velocity is

$$\omega = (10\,\tfrac{\text{rev}}{\text{min}})\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 1.05\,\tfrac{\text{rad}}{\text{s}}$$

so its angular momentum is

$$L = I\omega = (250\,\text{kg}\cdot\text{m}^2)(1.05\,\tfrac{\text{rad}}{\text{s}}) = 262\,\tfrac{\text{kg}\cdot\text{m}^2}{\text{s}}$$

The child is motionless, so this is the total initial angular momentum of the system: $L_i = 262\,\frac{\text{kg}\cdot\text{m}^2}{\text{s}}$.

After the child steps onto the merry-go-round (and stays at the same place near its outer edge) we have a rotating system with a *different* moment of inertia. The child is (basically) a point mass at a distance of 2.0 m from the rotation axis, so the *new* moment of inertia is found from summing the moments of the original wheel and the child:

$$I' = (250\,\text{kg}\cdot\text{m}^2) + M_{\text{child}}R^2 = (250\,\text{kg}\cdot\text{m}^2) + (25.0\,\text{kg})(2.0\,\text{m})^2 = 350\,\text{kg}\cdot\text{m}^2$$

and if ω' is the final angular velocity, then the final angular momentum is given by:

$$L_f = I'\omega'$$

But from angular momentum conservation, $L_i = L_f$ and this lets us solve for ω':

$$L_i = 262\,\tfrac{\text{kg}\cdot\text{m}^2}{\text{s}} = L_f = I'\omega' = (350\,\text{kg}\cdot\text{m}^2)\omega'$$

which gives

$$\omega' = \frac{(262\,\frac{\text{kg}\cdot\text{m}^2}{\text{s}})}{(350\,\text{kg}\cdot\text{m}^2)} = 0.748\,\tfrac{\text{rad}}{\text{s}}$$

We can convert this back to revolutions per minute to give:

$$\omega' = (0.748\,\tfrac{\text{rad}}{\text{s}})\left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right)\left(\frac{60\,\text{s}}{1\,\text{min}}\right) = 7.14\,\tfrac{\text{rev}}{\text{min}}\ .$$

The merry-go-round slows down to $7.14\,\frac{\text{rev}}{\text{min}}$ after the child steps on.

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12. Two astronauts (see Fig. 2.6) each having a mass of 75 kg are connected by a 10 m rope of negligible mass. They are isolated in space, orbiting their center of mass at speeds of $5.0\,\frac{\text{m}}{\text{s}}$. Calculate (a) the magnitude of the angular momentum of the system by treating the astronauts as particles and (b) the rotational energy of the system. By pulling on the rope, the astronauts shorten the distance between them to 5.0 m. (c) What is the new angular momentum of the system? (d) What are their new speeds? (e) What is the new rotational energy of the

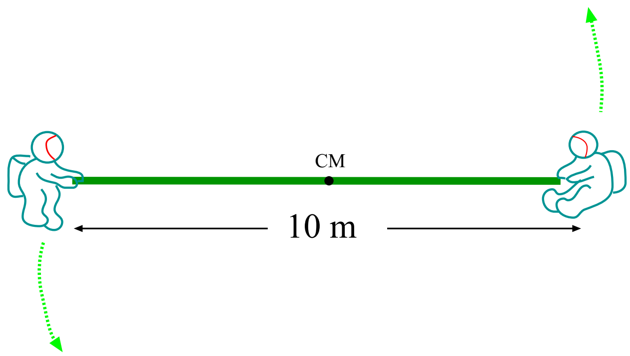

Figure 2.6: Two astronauts out in space, playing around with a rope, in Example 12.

system? (f) How much work is done by the astronauts in shortening the rope? [Ser4 11-53]

(a) Each astronaut in Fig. 2.6 has a mass of 75 kg, a speed of $5.0\,\frac{\mathrm{m}}{\mathrm{s}}$ and moves perpendicularly to the line which joins the origin (i.e. the CM) to their locations. Each one is 5.0 m from the origin, so the magnitude of the angular momentum of each astronaut is

$$\ell = rp\sin 90^\circ = (5.0\,\mathrm{m})(75\,\mathrm{kg})(5.0\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 1.88\times 10^3\,\tfrac{\mathrm{kg\cdot m^2}}{\mathrm{s}}$$

(By the right–hand–rule, the angular momentum vector for each one points *out of the page*. Since the rope is taken to be massless, the sum of the angular momenta for the astronauts is the **L** for the system; these just add together to give:

$$L = 2(1.88\times 10^3\,\tfrac{\mathrm{kg\cdot m^2}}{\mathrm{s}}) = 3.75\times 10^3\,\tfrac{\mathrm{kg\cdot m^2}}{\mathrm{s}}$$

(b) The rotational energy of the system is simply the total kinetic energy of the astronauts (again, the rope is massless). This is:

$$K = 2\left(\tfrac{1}{2}(75\,\mathrm{kg})(5.0\,\tfrac{\mathrm{m}}{\mathrm{s}})^2\right) = 1.88\times 10^3\,\mathrm{J} = 1.88\,\mathrm{kJ}$$

(c) As the astronauts pull on the rope to decrease their separation, there are all kinds of internal forces in the astronaut–rope system, but *there are no external torques on the system*. As a result, the total angular momentum stays the same: Its magnitude is still

$$L = 3.75\times 10^3\,\tfrac{\mathrm{kg\cdot m^2}}{\mathrm{s}}$$

(d) In the new configuration, each astronaut will be $5.0\,\mathrm{m}/2 = 2.5\,\mathrm{m}$ from the center of rotation and will still have a velocity (and linear momentum) perpendicular to the line joining the rotation center to his location. If the new linear momentum of each astronaut is now p', then we can use the expression for the total angular momentum in the new configuration to write:

$$L' = L = 3.75\times 10^3\,\tfrac{\mathrm{kg\cdot m^2}}{\mathrm{s}} = 2(r'p') = 2(2.5\,\mathrm{m})p'$$

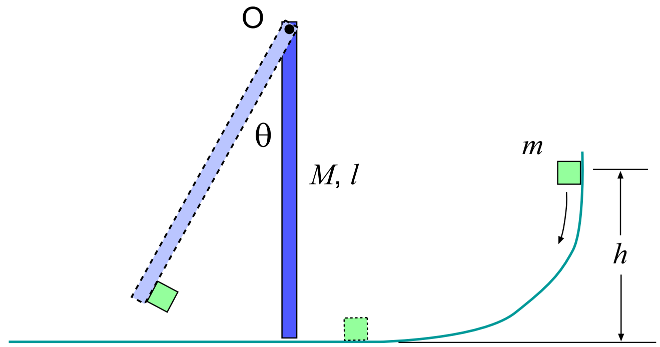

Figure 2.7: Odd contraption where a sliding particle sticks to the end of a rod which is pivoted about O, in Example 13.

and then solve for p':

$$p' = \frac{(3.75 \times 10^3 \, \frac{\text{kg}\cdot\text{m}^2}{\text{s}})}{2(2.5\,\text{m})} = 750 \, \frac{\text{kg}\cdot\text{m}}{\text{s}}$$

and from this we can get the new *speed* of each astronaut:

$$v' = \frac{p'}{m} = \frac{(750 \, \frac{\text{kg}\cdot\text{m}}{\text{s}})}{(75.0\,\text{kg})} = 10.0 \, \frac{\text{m}}{\text{s}}$$

(e) The total kinetic energy of the astronauts is now

$$K' = 2\left(\tfrac{1}{2}(75.0\,\text{kg})(10.0\,\tfrac{\text{m}}{\text{s}})^2\right) = 7.50 \times 10^3\,\text{J} = 7.50\,\text{kJ}$$

(f) Our answer for (e) is larger (in fact, quite a bit larger!) than the initial kinetic energy which we calculated in part (b). The difference had to come from the work done by astronauts in pulling on the rope. So the work done was

$$W = K' - K = 7.50\,\text{kJ} - 1.88\,\text{kJ} = 5.62\,\text{kJ} \; .$$

The astronauts did 5.62 kJ of work in shortening the length of rope between them.

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13. The particle of m in Fig. 2.7 slides down the frictionless surface and collides with the uniform vertical rod, sticking to it. The rod pivots about O through the angle θ before momentarily coming to rest. Find θ in terms of the other parameters given in the figure. [HRW5 12-69]

To make any sense of the motion of the small mass and the rod we need to go through this collision step–by–step to see which of physical principles can be applied.

The mass starts at a height h above the level part of the surface and slides down the smooth slope. Just before it hits the rod, we know that the mechanical energy of the system has been conserved because no friction–type force have been acting. This will allow us to find the speed of the mass at the bottom of the ramp.

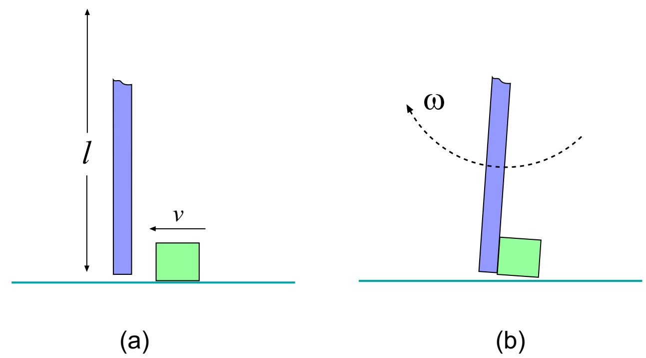

Figure 2.8: (a) Just before mass strikes end of rod it has linear motion, with speed v. (b) After mass sticks to rod, both rod and mass rotate about the pivot with angular velocity ω.

Next, the mass has a very brief (and very sticky) encounter with the end of the rod. We are used to treating a set of masses as being *isolated* during the short time that they exert forces on one another. And indeed during the collision the mass and rod *are* isolated, in a sense. But it is *not* true that we can ignore the external *forces* during their brief encounter, because in fact the pivot will exert a significant force on the rod that can't be ignored. But it is true that we can ignore the external *torques* that act on the rod–mass system (about the pivot, that is) because the pivot force will give no *torque*. And because of this we will be able to use the fact that the total angular momentum is conserved during the rapid, sticky collision of the mass and the end of the rod.

Finally, after the mass is stuck to the end of the rod, the rod makes a swing upward, momentarily coming to rest. During this part of the motion it is *total mechanical energy* which is conserved – as long as there is no friction in the pivot!

Now we start writing down some equations to express the principles laid out here.

First we apply the condition of energy conservation between the initial position of the mass and the instant before it hit the rod. At first, the mass has only gravitational energy: $E_i = mgh$. Just before the collision, it has only kinetic energy; if its speed there is v, then $E_f = \frac{1}{2}mv^2$ and energy conservation, $E_i = E_f$ gives us:

$$mgh = \tfrac{1}{2}mv^2 \qquad \Longrightarrow \qquad v = \sqrt{2gh} \tag{2.13}$$

Now we look at the rapid collision, in which *angular momentum* is conserved. Just before the mass m hits the end of the rod, its motion is horizontal (perpendicular to line joining the pivot and its location) and it has the speed v which we have already calculated, as shown in Fig. 2.8) (a). At this time its distance from the pivot is l, so its angular momentum about the pivot (in the clockwise sense) is $L_i = lmv$. Since at this time the rod is motionless, that is the *total* angular momentum of the mass–rod system.

After the collision the mass and rod form a rotating system with moment of inertia

$$I = \tfrac{1}{3}Ml^2 + ml^2 = \left(\frac{M}{3} + m\right) l^2$$

since the mass m is assumed small and is fixed at a distance l from the pivot. If the angular

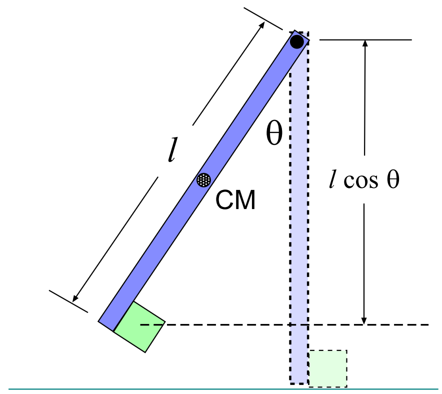

Figure 2.9: Some geometry for the "final" position of the mass/rod combination.

velocity of the rod/mass just after the collision is ω, then the final angular momentum is

$$L_f = I\omega = \left(\frac{M}{3} + m\right) l^2\omega$$

Conservation of angular momentum, $L_i = L_f$ gives us:

$$lmv = \left(\frac{M}{3} + m\right) l^2\omega \tag{2.14}$$

Finally, after the collision we have energy conservation all during the upward swing of the rod/mass. (We will now let the letters i and f refer to initial and final positions for *this* part.) Just after the mass has stuck to the rod, the kinetic energy of the system is its rotational kinetic energy,

$$K_i = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\left(\frac{M}{3} + m\right) l^2\omega^2 \ .$$

To calculate potential energies we will measure height from the "ground" level. The mass is small and basically has no height but the CM of the rod is at a height $\frac{l}{2}$ so the initial potential energy is

$$U_i = Mg\frac{l}{2}$$

At the top of the swing, we know what the final kinetic energy is: It's zero! As for the final potential energy we will need to do a little geometry, as in Fig. 2.9. From this figure we can see that the height of the small mass is now $l - l\cos\theta$. The vertical distance of the rod's CM downward from the pivot is $\frac{l}{2}\cos\theta$, so that its height is now $l - \frac{l}{2}\cos\theta$. Then the final potential energy is (after a little factoring):

$$U_f = mgl(1 - \cos\theta) + Mgl(1 - \tfrac{1}{2}\cos\theta)$$

The change in potential energy is

$$\Delta U \quad = \quad U_f - U_i = mgl(1 - \cos\theta) + Mgl(1 - \tfrac{1}{2}\cos\theta) - Mg\frac{l}{2}$$

$$
\begin{aligned}
&= mgl(1-\cos\theta) + Mgl(\tfrac{1}{2} - \tfrac{1}{2}\cos\theta) \\
&= mgl(1-\cos\theta) + \frac{Mgl}{2}(1-\cos\theta) \\
&= \left(m + \frac{M}{2}\right) gl(1-\cos\theta)
\end{aligned}
$$

and the change in kinetic energy is just

$$\Delta K = K_f - K_i = -\tfrac{1}{2}\left(\frac{M}{3} + m\right) l^2\omega^2$$

so conservation of energy, $\Delta K + \Delta U = 0$ gives us:

$$\left(m + \frac{M}{2}\right) gl(1-\cos\theta) - \tfrac{1}{2}\left(\frac{M}{3} + m\right) l^2\omega^2 = 0 \tag{2.15}$$

Have we done enough physics to get us to the answer? In the above equations, the unknowns are v, ω and θ, and we do have three equations: 2.13, 2.14 and 2.15. So we *can* solve for them; in particular we can find θ which is what the problem asks for. (The answer will be expressed in terms of M, m, and l which we take as given.)

Here's one way we can solve them. In Eq. 2.15 we will need ω^2, so we can write 2.14 as:

$$\omega = \frac{mv}{\left(\frac{M}{3} + m\right) l}$$

then square it and use $v^2 = 2gh$ from 2.13:

$$\omega^2 = \frac{m^2v^2}{\left(\frac{M}{3} + m\right)^2 l^2} = \frac{2m^2gh}{\left(\frac{M}{3} + m\right)^2 l^2}$$

Using this, we can substitute for ω^2 in Eq. 2.15 to get

$$\left(m + \frac{M}{2}\right) gl(1-\cos\theta) = \tfrac{1}{2}\left(\frac{M}{3} + m\right) l^2 \frac{2m^2gh}{\left(\frac{M}{3} + m\right)^2 l^2}$$

There is much to cancel here! Some algebra gives us:

$$(1-\cos\theta) = \frac{m^2h}{l\left(m + \frac{M}{2}\right)\left(\frac{M}{3} + m\right)}$$

Don't despair; we're nearly home! Multiplying the top and bottom of the right hand side by 6 tidies things up to give:

$$(1-\cos\theta) = \frac{6m^2h}{l(2m+M)(M+3m)}$$

Isolate $\cos\theta$ to get:

$$\cos\theta = 1 - \frac{6m^2h}{l(2m+M)(3m+M)}$$

And finally:

$$\theta = \cos^{-1}\left[1 - \frac{6m^2h}{l(2m+M)(3m+M)}\right]$$

Finished!